Name:

THE INSTITUTE OF EDUCATION

MATHEMATICS

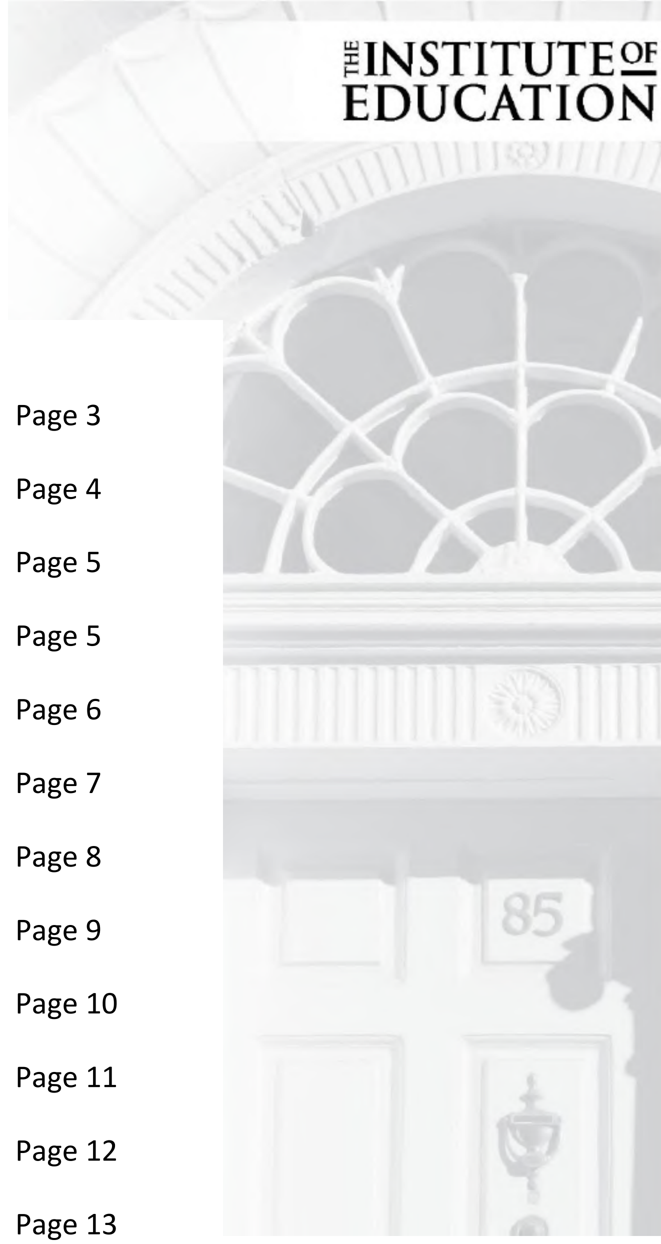
Hilary Dorgan

Leaving Certificate

Higher Level

2020 - 2021

CALCULUS NOTES



CONTENTS

THE INSTITUTE OF EDUCATION

Differential Calculus Paper 1 SYLLABUS

Functions and Calculus

5.2 Calculus		– find first and second derivatives of linear, quadratic and cubic functions by rule – associate derivatives with slopes and tangent lines – apply differentiation to • rates of change • maxima and minima • curve sketching	– differentiate linear and quadratic functions from first principles – differentiate the following functions • polynomial • exponential • trigonometric • rational powers • inverse functions • logarithms – find the derivatives of sums, differences, products, quotients and compositions of functions of the above form – apply the differentiation of above functions to solve problems – recognise integration as the reverse process of differentiation – use integration to find the average value of a function over an interval
5.2 Calculus (continued)			– integrate sums, differences and constant multiples of functions of the form • x^a, where $a \in \mathbb{Q}$ • a^x, where $a \in \mathbb{R}$ • Sin ax, where $a \in \mathbb{R}$ • Cos ax, where $a \in \mathbb{R}$ – determine areas of plane regions bounded by polynomial and exponential curves

Limits

Example 1: What is $\lim\limits_{x \to 5}(2x + 3)$?

This is equivalent to asking, "what happens the value of $2x + 3$ as the value of x gets closer and closer to 5"?

Solution 1: Substitute "5" for x, giving an answer $2(5) + 3 = 13$.

Example 2: What is $\lim\limits_{x \to 7} \frac{x-7}{x^2-49}$?

Solution 2: Substituting "7" for x, gives an ***indeterminate*** answer, $\frac{0}{0}$. (There is no answer to this.)

But we can factorise, giving us: $\lim\limits_{x \to 7} \frac{x-7}{(x-7)(x+7)} = \lim\limits_{x \to 7} \frac{1}{(x+7)} = \frac{1}{14}$

<u>The laws of Limits</u>: [Looks complicated, but all fairly obvious anyway, don't learn off or anything , just understand]

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$\lim_{x \to a}[f(x).\ g(x)] = \lim_{x \to a} f(x) .\ \lim_{x \to a} g(x)$$

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$$

Example 3: What is $\lim\limits_{x \to 25} \frac{x^2 - 625}{\sqrt{x}-5}$?

Solution 3: **(**Substituting "25" for x, gives an ***indeterminate*** answer, $\frac{0}{0}$. We need another method!)

$$\lim_{x \to 25} \frac{x^2-625}{\sqrt{x}-5} = \lim_{x \to 25} \frac{(x-25)(x+25)\,(\sqrt{x}+5)}{(\sqrt{x}-5)\ \ (\sqrt{x}+5)}$$ [Multiplying above and below by conjugate of below]

$$= \lim_{x \to 25} \frac{(x-25)(x+25)\,(\sqrt{x}+5)}{x-25}$$ [Denominator is equal to the difference of two squares]

$$= \lim_{x \to 25} \frac{(x+25)\,(\sqrt{x}+5)}{1} = \frac{(25+25)\,(\sqrt{25}+5)}{1} = (50)(10) = 500.$$

It is interesting to check limit results by substituting a value very close to the given value, e.g. 24.9 or 24.99 in this case.

Using 24.9: $\frac{24.9^2-625}{\sqrt{24.9}-5} = \frac{-4.99}{-0.01001} = 496.5.$ Using 24.99: $\frac{24.99^2-625}{\sqrt{24.99}-5} = \frac{-0.4999}{-0.0010001} = 499.85$

Example 4: What is $\lim\limits_{x \to 1} \frac{x^2}{(8-12x)-(2-x)^3}$?

Solution 4: $\lim\limits_{x \to 1} \frac{x^2}{(8-12x)-(2-x)^3} = \lim\limits_{x \to 1} \frac{x^2}{(8-12x)-(8-12x+6x^2-x^3)} = \lim\limits_{x \to 1} \frac{x^2}{x^3-6x^2} = \lim\limits_{x \to 1} \frac{1}{x-6} = \frac{-1}{5}$

[***Note***: Substituting 0.99 for *x*, $\frac{0.99^2}{(8-12(0.99))-(2-0.99)^3} = \frac{0.9801}{-4.910} = -0.1996 \approx -0.2$]

Trigonometric Limits

θ (in degrees)	θ (in radians)	Sin θ	$\frac{\text{Sin }\theta}{\theta}$
10°	0.1745329252	0.1736481777	0.9949307662
1°	0.01745329252	0.01745240644	0.9999492312
0.1°	0.001745329252	0.001745328366	0.9999994923
0.01°	0.0001745329252	0.0001745329243	0.9999999949
0.001°	0.00001745329252	0.00001745329252	0.9999999999

This table shows that as the angle θ (in radians) gets smaller and smaller and closer and closer to 0, then the ratio of the Sine of the angle to the angle gets closer and closer to 1.
(This topic is also dealt with in Trigonometry notes (see *page 19*, *Trigonometry Notes*.)

Mathematically, we write this important result as: $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.}$

Differentiation from First Principles

Example 5: Find $\frac{dy}{dx}$ from first principles if $y = 2x^2 + 5x$.

Solution 5: $f(x) = 2x^2 + 5x$

$$f(x+h) = 2(x+h)^2 + 5(x+h)$$

$$= 2x^2 + 4hx + 2h^2 + 5x + 5h$$

$$\Rightarrow f(x+h) - f(x) = (2x^2 + 4hx + 2h^2 + 5x + 5h) - (2x^2 + 5x)$$

$$= 4hx + 2h^2 + 5h$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = 4x + 2h + 5$$ [Dividing both sides of equation by h]

But $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (4x + 2h + 5) = 4x + 5$

The Rules of Differentiation

Example 6: Prove by induction that if $f(x) = x^n$, then $\frac{df}{dx} = n\,x^{n-1}$ for $n \in N$.

Solution 6: (i) <u>If $n = 1$</u>: If $f(x) = x^1$, is $\frac{df}{dx} = 1\,x^{1-1} = x^0 = 1$? (i.e. Is the differential of x = 1?)

We know that the differential of x is 1 and therefore the proposition is true for $n = 1$.
[But, this would have to be proved separately using first principles. Why?]

(ii) We now assume that the proposition is true for $n = k$.

<u>If $n = k$</u>: If $f(x) = x^k$, then $\frac{df}{dx} = kx^{k-1}$.

We must now prove that the proposition is true for $n = k+1$.

<u>If $n = k+1$</u>: **To prove**: If $f(x) = x^{k+1}$, then $\frac{df}{dx} = (k+1)\,x^k$.

$f(x) = x^{k+1} = (x)(x)^k$

Using product rule: LHS = $\frac{df}{dx} = (x)(kx^{k-1}) + (x)^k\,1$ [Using proposition ***k***, already assumed]

$= kx^k + x^k = x^k\,(k+1) = (k+1)x^k$ = RHS

(iii) But, proposition 1 is true and therefore, by induction, the proposition is true for all $n \in N$.

The general rule for differentiation in example 6 above (if $f(x) = x^n$, then $\frac{df}{dx} = n\,x^{n-1}$), is in fact true for $n \in R$ (but we only need to be able to prove it for $n \in N$).

It is a really important rule and we use it very often to differentiate functions by rule, rather than by first principles, which is far too tedious, unless specifically asked for.

For instance, if $f(x) = 7$ then it can also be said that $f(x) = 7x^0 \quad \Rightarrow \quad \frac{df}{dx}\ OR\ f'(x) = 0\,(7)(x)^{-1} = 0$

Thus, the $\frac{d}{dx}(constant) = 0\ \ ALWAYS.$

If $f(x) = u(x) + v(x)$, then $\dfrac{df}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ [The differential of a sum is the sum of the differentials of each]

If $f(x) = u(x)v(x)$, then $\dfrac{df}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ [The differential of a Product]

Differential of a Product $= First \times Differential\ of\ Second \ + \ Second \times Differential\ of\ First$

If $f(x) = \frac{u(x)}{v(x)}$, then $\dfrac{df}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ [The differential of a Quotient]

Differential of a Quotient $= \frac{(Bottom)\,(Differential\ of\ Top)\ -\ (Top)\,(Differential\ of\ Bottom)}{(Bottom)^2}$

The **CHAIN rule** of differentiation (next page): If $y = [f(x)]^n \ \Rightarrow \ \frac{dy}{dx} = n\,[f(x)]^{n-1}\,[f'(x)]$ [Chain Rule]

PRACTISING the Rules of Differentiation

Example 7: If $y = x^2(7x^3 - 2x + 3)$, find $\frac{dy}{dx}$. [The Product Rule]

Solution 7:

$$\frac{dy}{dx} = \text{(First)} \times \text{(Differential of Second)} + \text{(Second)} \times \text{(Differential of First)}$$
$$= (x^2)\ (21x^2 - 2) + (7x^3 - 2x + 3)(2x)$$
$$= 21x^4 - 2x^2 + 14x^4 - 4x^2 + 6x$$
$$= 35x^4 - 6x^2 + 6x$$

Alternative: $y = x^2(7x^3 - 2x + 3) \Rightarrow y = 7x^5 - 2x^3 + 3x^2$

$$\Rightarrow \frac{dy}{dx} = 35x^4 - 6x^2 + 6x$$

Example 8: If $y = \frac{5x^2 - 7x}{2x + 11}$, find $\frac{dy}{dx}$. [The Quotient Rule]

Solution 8:

$$\frac{dy}{dx} = \frac{(Bottom)\,(Differential\ of\ Top) - (Top)\,(Differential\ of\ Bottom)}{(Bottom)^2}$$

$$= \frac{(2x + 11)\,(10x - 7) - (5x^2 - 7x)\,(2)}{(2x + 11)^2} = \frac{20x^2 - 14x + 110x - 77 - 10x^2 + 14x}{(2x + 11)^2}$$

$$= \frac{10x^2 + 110x - 77}{(2x + 11)^2}$$

The CHAIN rule of differentiation: If $y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n\,[f(x)]^{n-1}\,[f'(x)]$

Example 9: Differentiate (i) $\sqrt{7x - 11}$, (ii) $\sqrt[3]{\frac{x^2-1}{2x+1}}$

Solution 9: (i) $f(x) = \sqrt{7x - 11} = (7x - 11)^{\frac{1}{2}}$ [Always re-write question in calculus friendly form]

$$\Rightarrow f'(x) = \frac{1}{2}\,(7x - 11)^{\frac{-1}{2}}\,(7)$$ ["7" is the differential of what is in the brackets]

$$= \frac{7}{2\sqrt{7x-11}}$$

(ii) $f(x) = \sqrt[3]{\frac{x^2 - 1}{6x + 1}} = (\frac{x^2 - 1}{6x + 1})^{\frac{1}{3}}$ [Always re-write question in calculus friendly form]

$$\Rightarrow f'(x) = \frac{1}{3}\left(\frac{x^2-1}{6x+1}\right)^{\frac{-2}{3}} \times \text{(differential of what is in the brackets)}$$

$$= \frac{1}{3}\left(\frac{x^2 - 1}{6x + 1}\right)^{\frac{-2}{3}} \left[\frac{(6x+1)\,(2x) - (x^2-1)(6)}{(6x+1)^2}\right]$$

$$= \frac{1}{3}\left(\frac{x^2 - 1}{6x + 1}\right)^{\frac{-2}{3}} \left[\frac{12x^2 + 2x - 6x^2 + 6}{(6x+1)^2}\right]$$

$$= \frac{1}{3}\left(\frac{x^2 - 1}{6x + 1}\right)^{\frac{-2}{3}} \left[\frac{6x^2 + 2x + 6}{(6x + 1)^2}\right] = \frac{2}{3}\left(\frac{x^2 - 1}{6x + 1}\right)^{\frac{-2}{3}} \left[\frac{3x^2 + x + 3}{(6x + 1)^2}\right]$$

Differentiating trigonometric functions

Example 10: If $y = sin\ (x^3 - 5x^2 + 3)$, find $\frac{dy}{dx}$.

Solution 10: $\frac{dy}{dx} = cos\ (x^3 - 5x^2 + 3)\,(3x^2 - 10x) = (3x^2 - 10x)\ \ cos\ (x^3 - 5x^2 + 3)$.

Example 11: If $y = \frac{1 - cos\,2x}{7 + sin\,3x}$, find $\frac{dy}{dx}$ at $x = 0$.

Solution 11: $\frac{dy}{dx} = \frac{(7 + sin\,3x)(2\,sin\,2x)\ -\ (1 - cos\,2x)\,(3\,cos\,3x)}{(7 + sin\,3x)^2}$

$\frac{dy}{dx}$ at $x = 0$: $\frac{(7+0)(0)\ -\ (0)(3)}{49} = 0$

Example 12: If $f(x) = tan\frac{5x}{3x-1}$ find $f'(x)$ at $x = 1$.

Solution 12: $f'(x) = sec^2\left(\frac{5x}{3x-1}\right) \times \frac{(3x-1)\,(5) - (5x)\,(3)}{(3x-1)^2}$

$$\Rightarrow f'(1) = sec^2\left(\frac{5}{3-1}\right) \times \frac{(3-1)\,(5) - (5)\,(3)}{(3-1)^2} = sec^2\left(\frac{5}{2}\right) \times \frac{10-15}{4} = \frac{-5}{4}\ sec^2\left(\frac{5}{2}\right) \approx -0.80$$

Differentiating exponential functions

Example 13: If $f(x) = 17e^{-2x^2}$ find $f'(x)$.

Solution 13: $f'(x) = \left(17\,e^{-2x^2}\right) \times (-4x) = -68xe^{-2x^2}$.

Example 14: If $y = x^5\,e^{-3x}$ find $\frac{dy}{dx}$ at $x = 2$.

Solution 14: $\frac{dy}{dx} = (x^5)\,(e^{-3x})\,(-3) + (e^{-3x})\,(5x^4) = e^{-3x}(-3x^5 + 5x^4)$

$\frac{dy}{dx}$ at $x = 2$: $e^{-6}(-3(2)^5 + 5(2)^4) = e^{-6}(-96 + 80) = {}^{-16}/_{e^6}$

Example 15: Differentiate ${}^{e^{2x}}/_{3x^2}$ with respect to x.

Solution 15: $f'(x) = \frac{(3x^2)\,(2\,e^{2x}) - (e^{2x})(6x)}{(3x^2)^2} = \frac{(6x^2 - 6x)(e^{2x})}{9x^4}$

Differentiating logarithmic functions

Example 16: If $y = log_e(2\ sin\,3x)$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{12}$.

Solution 16: $\frac{dy}{dx} = \frac{f'(x)}{f(x)} = \frac{2\,(3)\,cos\,3x}{2\,sin\,3x} = \frac{3}{tan\,3x}$

$\frac{dy}{dx}$ at $x = \frac{\pi}{12}$: $\frac{3}{tan\frac{\pi}{4}} = \frac{3}{1} = 3$

Example 17: If $f(x) = [ln\,(13x - 5)]^3$, find (i) $f'(x)$ and (ii) $f'(2)$ correct to two places of decimal.

Solution 17: (i) $f'(x) = 3\,[ln\,(13x - 5)]^2\left(\frac{13}{(13x-5)}\right) = \frac{39\,[ln\,(13x-5)]^2}{13x-5}$

(ii) $f'(2) = \frac{39\,[ln\,(26-5)]^2}{26-5} = \frac{39\,[ln\,(21)]^2}{21} = \frac{13\,(3.044522)^2}{21} = \frac{13\,(9.26912)}{21} \approx 5.74$

Example 18: Differentiate $log_e \sqrt{\frac{2x^2}{x^2+1}}$

Solution 18: $y = log_e \sqrt{\frac{2x^2}{x^2+1}} \Rightarrow y = \frac{1}{2} log_e \frac{2x^2}{x^2+1} = \frac{1}{2}[log_e(2x^2) - log_e(x^2+1)]$

[Using rules of logs]

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{4x}{2x^2} - \frac{2x}{x^2+1}\right] = \frac{1}{2}\left[\frac{2}{x} - \frac{2x}{x^2+1}\right] = \frac{1}{x} - \frac{x}{x^2+1}$$

$$= \left[\frac{(1)(x^2+1) - (x)(x)}{(x)(x^2+1)}\right]$$

$$= \frac{(x^2+1) - (x^2)}{(x)(x^2+1)}$$

$$= \frac{1}{(x)(x^2+1)}$$

Differentiating inverse trigonometric functions like $sin^{-1} x$ and $tan^{-1} x$

Page 25 of the Mathematical Tables tell us that if $f(x) = sin^{-1}\frac{x}{a}$ then $f'(x) = \frac{1}{\sqrt{a^2 - x^2}}$, where a is a constant.

It also tells us that if $f(x) = tan^{-1}\frac{x}{a}$ then $f'(x) = \frac{1}{a^2 + x^2}$, where a is a constant.

$$\frac{d}{dx}\left(sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$$

If $y = sin^{-1} f(x)$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-[f(x)]^2}} \times f'(x)$

AND

$$\frac{d}{dx}\left(tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$$

If $y = tan^{-1} f(x)$, $\frac{dy}{dx} = \frac{1}{1 + [f(x)]^2} \times f'(x)$

Example 19: If $(x) = tan^{-1}\frac{x}{3}$, find $f'(x)$.

Solution 19: <u>**Method 1**</u>: $f'(x) = \frac{3}{3^2 + x^2} = \frac{3}{9 + x^2}$ [Because x has been replaced by "3"]

<u>**Method 2**</u>: $f'(x) = \frac{1}{1 + (\frac{x}{3})^2} \times \frac{1}{3} = \frac{9}{9 + x^2} \times \frac{1}{3} = \frac{3}{9 + x^2}$

Example 20: If $(x) = (sin^{-1} 3x)^5$, find $f'(x)$.

Solution 20: $f'(x) = 5(sin^{-1} 3x)^4 \times \frac{1}{\sqrt{1-(3x)^2}} \times 3 = \frac{15(sin^{-1} 3x)^4}{\sqrt{1-(3x)^2}}$

Example 21: If $(x) = tan^{-1}\left(\frac{3x}{2x-7}\right)$, show that $f'(0) = \frac{-3}{7}$.

Solution 21: $f'(x) = \frac{1}{1+(\frac{3x}{2x-7})^2} \times \frac{(2x-7)(3)-(3x)(2)}{(2x-7)^2} = \frac{(2x-7)^2}{(2x-7)^2+(3x)^2} \times \frac{6x-21-6x}{(2x-7)^2}$

$$= \frac{-21}{(2x-7)^2+(3x)^2} = \frac{-21}{4x^2-28x+49+9x^2}$$

$$= \frac{-21}{13x^2-28x+49}$$

$$\Rightarrow f'(0) = \frac{-21}{49} = \frac{-3}{7}$$

Second Derivatives

If $y = f(x)$, then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$.

Example 22: If $y = e^{4x}$, show that $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$.

Solution 22: $\frac{dy}{dx} = \frac{d}{dx}(e^{4x}) = 4e^{4x}$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4e^{4x}) = 16e^{4x}.$$

$$\therefore LHS = \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y$$

$$= 16e^{4x} - (8)(4e^{4x}) + 16(e^{4x})$$

$$= e^{4x}(16 - 32 + 16)$$

$$= (e^{4x})\,0 = 0 = RHS$$

Example 23: Given that $y = xe^{3x}$, find the value of $\left(\frac{dy}{dx}\right)^2 - e^{3x}\frac{d^2y}{dx^2}$ when $x = 0$.

Solution 23: $y = xe^{3x} \quad \Rightarrow \quad \frac{dy}{dx} = (x)(3e^{3x}) + (e^{3x})(1)$

$$= e^{3x}(3x+1)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}[e^{3x}(3x+1)]$$

$$= (e^{3x})(3) + (3x+1)(3e^{3x})$$

$$= (3e^{3x})(1 + 3x + 1)$$

$$= (3e^{3x})(3x+2)$$

$$\therefore \left(\frac{dy}{dx}\right)^2 - e^{3x}\frac{d^2y}{dx^2} = [e^{3x}(3x+1)]^2 - (e^{3x})(3e^{3x})(3x+2)$$

$$\therefore \left(\frac{dy}{dx}\right)^2 - e^{3x}\frac{d^2y}{dx^2} \; at\; x = 0:$$

$$= [1(1)]^2 - (1)(3)(2)$$

$$= 1 - 6 = -5$$

APPLICATIONS of DIFFERENTIATION

Finding the slope and equation of a tangent to a curve

A vital application of differentiation is the following:

$\boxed{\frac{dy}{dx} \text{ is the slope of a tangent to a curve at any point on the curve.}}$

To find the equation of a tangent to a curve at a given point $(x_1,\ y_1)$ on the curve, take the following steps:

1) Find $\frac{dy}{dx}$
2) Evaluate $\frac{dy}{dx}$ at $x = x_1$ [This gives m, the slope of the tangent to the curve at $(x_1,\ y_1)$]
3) Use equation $y - y_1 = m\,(x - x_1)$ to find the equation of the tangent to the curve at $(x_1,\ y_1)$

Example 24: Find the equation of the tangent to the curve $y = 10 - \cos x$ at the point $(\frac{\pi}{2}\ ,\ 0)$

Solution 24: $y = 10 - \cos x$

$\Rightarrow \frac{dy}{dx} = 0 + \sin x$

$\frac{dy}{dx}\ @x = \frac{\pi}{2} = \sin\frac{\pi}{2} = 1$ [This gives m, the slope of the tangent to the curve at $(\frac{\pi}{2}\ ,\ 0)$]

$\Rightarrow$ Eqn of tangent: $y - y_1 = m\,(x - x_1)$

$\Rightarrow$ Eqn of tangent: $y - 0 = 1\left(x - \frac{\pi}{2}\right) \quad \Rightarrow \quad \boxed{y = x - \frac{\pi}{2}}$

Example 25: The equation of a curve is $y = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in Z$.
The curve passes through the points $(0,\ 15)$ and $(2, -15)$ and at these points the slopes of the tangents are both -13.

(i) Find the values of $a, b, c,$ and d

(ii) Show that the graph cuts the $x - axis$ at the point $x = 5$.

(iii) Find the coordinates of the other two points where the curve cuts the $x - axis$.

(iv) Hence, sketch the curve.

Solution 25: **(i)** $(0,\ 15) \in\ y = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad 15 = 0 + 0 + 0 + d \quad \Rightarrow \quad \boldsymbol{d = 15}$

$(2, -15) \in\ y = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad -15 = 8a + 4b + 2c + 15$

$\Rightarrow \quad 8a + 4b + 2c = -30$

$\Rightarrow \quad 4a + 2b + c = -15$ … eqn 1

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$\frac{dy}{dx}$ @$x = 0$: $0 + 0 + c = -13 \Rightarrow \boldsymbol{c = -13}$

$\frac{dy}{dx}$ @$x = 2$: $3a(2)^2 + 2b(2) + (-13) = 12a + 4b + (-13) = -13$

$\Rightarrow 12a + 4b = 0 \Rightarrow 3a + b = 0$ ... eqn 2

From eqn 1: $4a + 2b - 13 = -15 \Rightarrow 2a + b = -1$

Using $3a + b = 0$ and $2a + b = -1$

$\Rightarrow \boldsymbol{-a = -1} \Rightarrow \boldsymbol{a = 1}$ **and** $\boldsymbol{b = -3}$

(ii) **If $x = 5$**: $y = ax^3 + bx^2 + cx + d \Rightarrow y = 1x^3 - 3x^2 - 13x + 15$

$\Rightarrow$ **If $x = 5$**: $\Rightarrow y = 1(5)^3 - 3(5)^2 - 13(5) + 15 = 125 - 75 - 65 + 15 = 140 - 140 = 0$

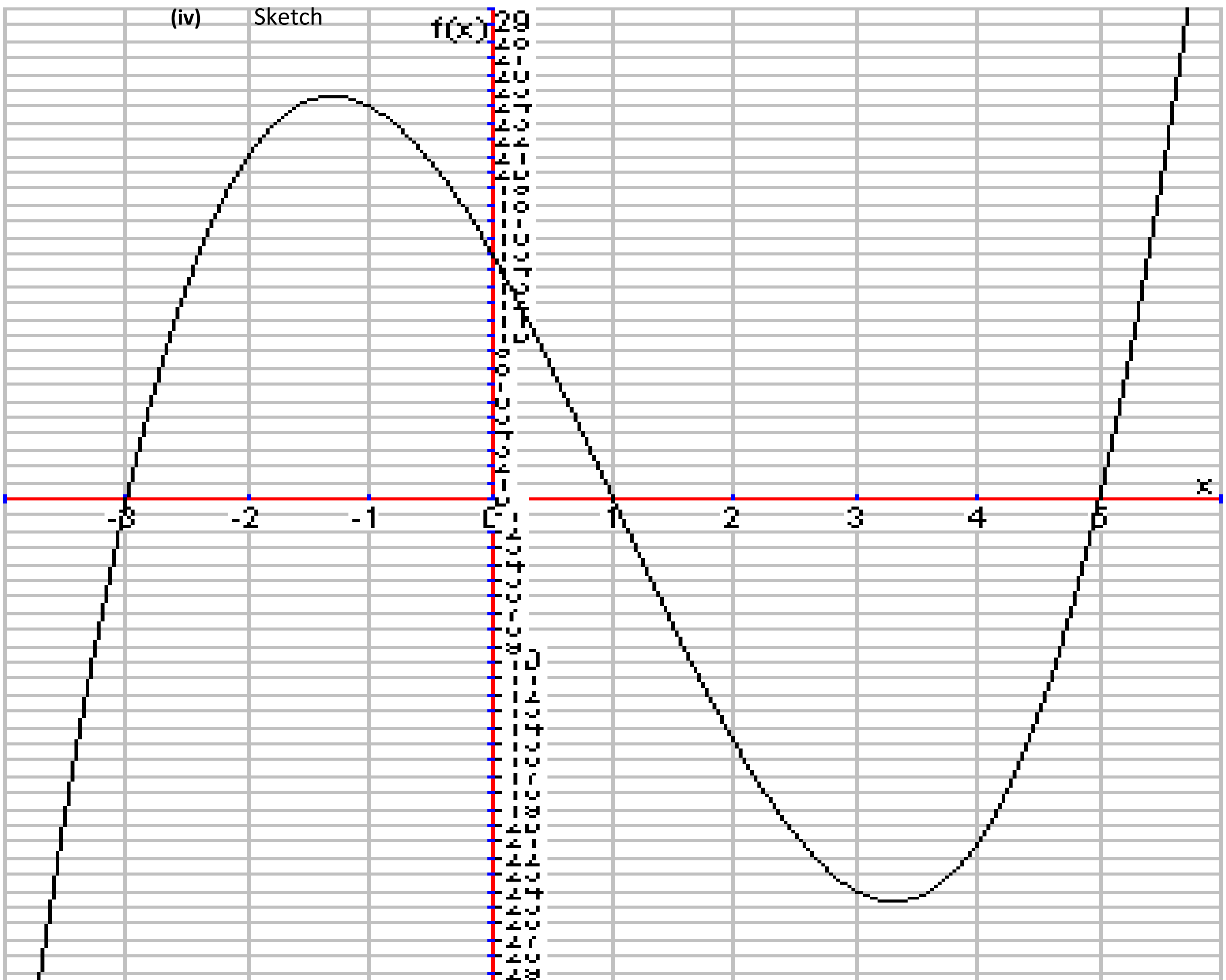
$\Rightarrow$ the graph cuts the $x - axis$ at $(5, 0)$.

(iii) $\boldsymbol{f(5) = 0 \Rightarrow x - 5}$ **is a factor**

$(x^3 - 3x^2 - 13x + 15) \div (x - 5) = x^2 + 2x - 3 = (x - 1)(x + 3)$

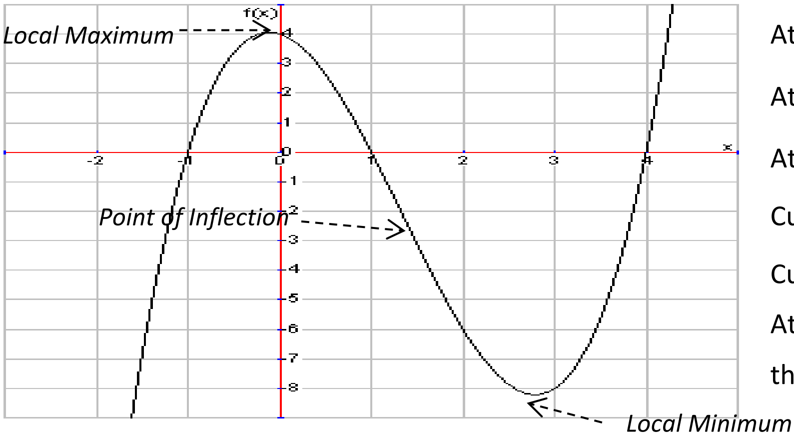
$\Rightarrow$ other two points where the curve cuts the $x - axis$: $(1, 0)$ and $(-3, 0)$.

(iv) Sketch



THE INSTITUTE OF EDUCATION

Finding maxima, minima and points of inflection of a curve



At ***local maximum***: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0.$

At ***local minimum***: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0.$

At ***point of inflection***: $\frac{d^2y}{dx^2} = 0.$

Curves are ***increasing*** when $\frac{dy}{dx} > 0.$

Curves are ***decreasing*** when $\frac{dy}{dx} < 0.$

At turning points, the slope of the tangent to the curve is parallel to the *x*-axis, i.e. $\frac{dy}{dx} = 0.$

Example 26: Find the local maximum, the local minimum and a point of inflection of the curve $y = -x^3 + 6x^2 - 9x + 3$. Use this information to sketch the curve.

Solution 26: If $y = -x^3 + 6x^2 - 9x + 3$

$\Rightarrow \quad \frac{dy}{dx} = -3x^2 + 12x - 9$

To find turning points, we must solve $\frac{dy}{dx} = 0$

$\therefore \quad -3x^2 + 12x - 9 = 0 \quad \Rightarrow \quad x^2 - 4x + 3 = 0$

$\therefore \quad (x-3)(x-1) = 0$

$\Rightarrow \quad x = 3 \ \ or \ \ x = 1 \quad \Rightarrow$ Turning points @ (3, 3) & (1, –1)

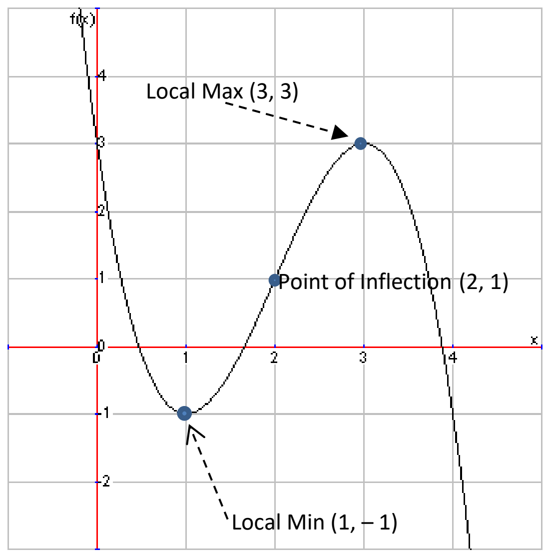
$\frac{d^2y}{dx^2} = -6x + 12$ [*y* coordinate is calculated from original equation]

$\frac{d^2y}{dx^2}$@ $x = 3$: $-6(3) + 12 = -6 < 0 \ \Rightarrow$ Max (3, 3)

$\frac{d^2y}{dx^2}$@ $x = 1$: $-6(1) + 12 = +6 > 0 \ \Rightarrow$ Min (1, –1)

If $\frac{d^2y}{dx^2} = 0$: $\Rightarrow \ -6x + 12 = 0 \quad \Rightarrow \quad x = 2\,,\ y = 1$

$\therefore$ Point of inflection @ (2, 1)



Example 27: Find the coordinates of the one turning point of the curve $y = -e^{2x} + 2e^x$ and determine whether this turning point is a maximum or a minimum.

Solution 27: If $y = -e^{2x} + 2e^x$, then $\frac{dy}{dx} = -2e^{2x} + 2e^x$.

Turning point when $\frac{dy}{dx} = -2e^{2x} + 2e^x = 0 \quad \Rightarrow \quad 2e^{2x} - 2e^x = 0.$

$\Rightarrow \ e^{2x} - e^x = 0 \quad \Rightarrow \quad e^x(e^x - 1) = 0$

$\Rightarrow \ e^x = 0$ or $e^x = 1$

$e^x = 0$ has no solution and if $e^x = 1$, then $x = 0$.

$\therefore$ Turning point at (0, 1) [If $x = 0,\ y = -e^{2(0)} + 2e^{(0)} = -1 + 2 = 1$]

$\frac{d^2y}{dx^2} = -4e^{2x} + 2e^x$

$\therefore \ \frac{d^2y}{dx^2}$ @ $x = 0$: $-4e^{2(0)} + 2e^{(0)} = -4 + 2 = -2 < 0 \ \Rightarrow$ turning point (0, 1) is a Maximum.

Sketching Curves with the help of calculus

Example 28: $y = \frac{x-1}{x-2}$ is a curve.

(i) Find the equation of the vertical asymptote of this curve.

(ii) Find the equation of the horizontal asymptote of this curve.

(iii) Prove that the curve has no turning points

(iv) Sketch the curve.

(v) If the slopes of the tangents to this curve at $(x_1,\ y_1)$ and $(x_2,\ y_2)$ are parallel and $x_1 \neq x_2$, prove that $x_1 + x_2 = 4$.

Solution 28:

(i) Vertical asymptote is found by letting the denominator = 0.

$\therefore\ x - 2 = 0 \ \Rightarrow\ \boldsymbol{x = 2}$ [This is the equation of ***vertical*** asymptote].

(ii) Horizontal asymptote is found by letting $y = \lim\limits_{x\to\infty} f(x)$.

$y = \lim\limits_{x\to\infty} f(x) \ \Rightarrow\ y = \lim\limits_{x\to\infty} \frac{x-1}{x-2}$.

To find the limit as $x \to \infty$, divide above and below by highest power of x.

$\therefore\ y = \lim\limits_{x\to\infty} \frac{1 - {}^{1}\!/_{x}}{1 - {}^{2}\!/_{x}} = \frac{1-0}{1-0} = 1$

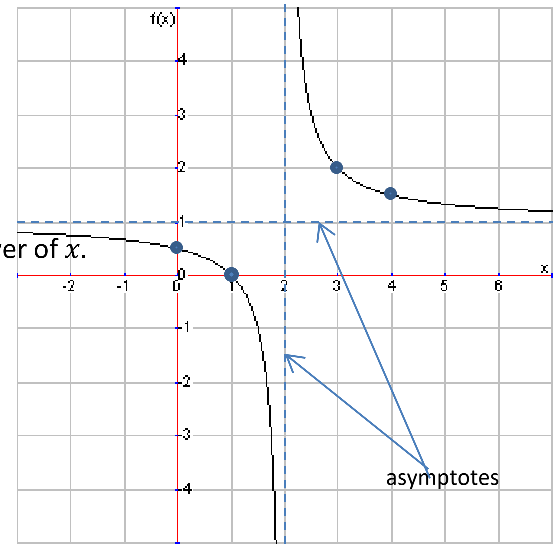
$\therefore$ Equation of ***horizontal*** asymptote: $\boldsymbol{y = 1}$.

(iii) $\frac{dy}{dx} = \frac{(x-2)(1)-(x-1)(1)}{(x-2)^2} = \frac{-1}{(x-2)^2}$

There is no value of x which will make $\frac{dy}{dx} = 0 \ \Rightarrow$ No turning points.

(iv) Find coordinates of points to the left and right of $x = 2$.

$(0,\ \frac{1}{2}), (1,\ 0)\ (3,\ 2)$ and $(4,\ \frac{3}{2})$ are on the curve.



[Graph paper should be used when sketching the curve (see graph)]

(v) The slope $(m_1)$of the tangent to the curve at $(x_1,\ y_1) = f'(x_1)$. But, $f'(x_1) = \frac{-1}{(x_1-2)^2}$.

The slope (m_2) of the tangent to the curve at $(x_2,\ y_2) = f'(x_2)$. But, $f'(x_2) = \frac{-1}{(x_2-2)^2}$.

But, $(m_1) = (m_2) \ \Rightarrow\ (x_1 - 2)^2 = (x_2 - 2)^2$ [But, $x_1 \neq x_2 \ \Rightarrow\ (x_1 - 2) \neq (x_2 - 2)$]

$\therefore\ x_1 - 2 = -(x_2 - 2) \ \Rightarrow\ x_1 + x_2 = 4$.

Displacement, Velocity and Acceleration

We often use the letter s to represent displacement (distance travelled in a certain direction).

Velocity $(v)$is the rate of change of displacement with respect to time.

In calculus, we can write the previous sentence as $v = \frac{ds}{dt}$.

Similarly, acceleration (a) is the rate of change of velocity with respect to time.

In calculus, we can write the previous sentence as $a = \frac{dv}{dt}$.

Thus, $a = \frac{dv}{dt} = \frac{d^2s}{d\,t^2}$ or, in English, acceleration is the second differential of displacement with respect to time.

Note: In calculus, if the "rate of change" of something is mentioned and there is no "with respect to …", then it can always be assumed that it is with respect to "time". That is, "time" is the default if nothing else is mentioned.

Example 29: An object moves in a straight line such that its distance from a fixed point, A , is given by $s = t^3 - 12t^2 + 21t + 5$, where $\boldsymbol{s}$ is in metres and $\boldsymbol{t}$ is in seconds.

(i) Find the displacement from the point A at the start of the motion.

(ii) Find the greatest displacement of the object from A (i.e. find local maximum)

(iii) When does the object come to instantaneous rest?

(iv) Find the speed of the object when $t = 10$ seconds.

(v) Find the minimum velocity of the object.

(vi) Find the acceleration of the particle after five seconds.

Solution 29:

(i) Finding $s, @\ t = 0$:

$$s = t^3 - 12t^2 + 21t + 5$$

$\therefore\ s\ @\ t = 0: \quad = (0)^3 - 12(0)^2 + 21(0) + 5 = 5$ m

(ii) Displacement is greatest when $\frac{ds}{dt} = 0$

$\frac{ds}{dt} = 3t^2 - 24t + 21 = 0$

$\Rightarrow\ t^2 - 8t + 7 = 0$

$\Rightarrow\ (t - 7)(t - 1) = 0$

$\Rightarrow\ t = 7$ or $t = 1$ s

Note: $\frac{d^2s}{dt^2} = 6t - 24$

At $t = 7$: $\frac{d^2s}{dt^2} = 6(7) - 24 = 18$ m/s $> 0 \Rightarrow$ Minimum displacement at $t = 7$

At $t = 1$: $\frac{d^2s}{dt^2} = 6(1) - 24 = -18$ m/s $< 0 \Rightarrow$ Maximum displacement at $t = 1$

When $t = 1$, $s = (1)^3 - 12(1)^2 + 21(1) + 5 = 15$ m

(iii) Instantaneous rest $\Rightarrow v = 0$.

As in part (ii) above, $v = \frac{ds}{dt} = 0$ when $t = 1$ and when $t = 7$ s.

(iv) Finding $v, @\ t = 10$

$$v = \frac{ds}{dt} = 3t^2 - 24t + 21$$

$\Rightarrow v, @\ t = 10$: $3(10)^2 - 24(10) + 21 = 81$ m/s

(v) Velocity is minimum (or lowest) when $\frac{dv}{dt} = \frac{d^2s}{dt^2} = 6t - 24 = 0$

$\Rightarrow$ Velocity is minimum when $t = 4$ s

[obviously a minimum rather than a maximum here as the second differential $= +6 > 0$]

$$v = 3t^2 - 24t + 21$$

$\Rightarrow\ v_{min} = 3(4)^2 - 24(4) + 21 = -27$ m/s

(vi) Acceleration $(a) = \frac{dv}{dt} = 6t - 24$

$\Rightarrow a, @\ t = 5 = 6(5) - 24$

$\Rightarrow a = 6$ m/s^2

Rates of Change

There are many examples in life where one quantity changes with another. The rate of how one quantity (like sale of ice creams) varies with another (like temperature) is a very important application of Calculus.
Many problems on rates of change involve displacement, velocity and acceleration that we have seen above.

Velocity (or speed) is the rate of change of displacement, i.e. $v = \frac{ds}{dt}$. It is measured in m/s.

Acceleration is the rate of change of velocity, i.e. $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. It is measured in m/s^2.

Example 30: A ball is fired straight up in the air. The height, h, of the ball above the ground is given by $h = 30t - 5t^2$ where t is the time in seconds after the ball was fired.

(i) After how many seconds does the ball hit the ground?

(ii) Find the speed of the ball after 2 seconds.

(iii) Find the maximum height reached by the ball.

(iv) Find the constant deceleration of the ball.

(v) When is the speed of the ball equal to 15 m/s?

Solution 30: **(i)** When the ball hits the ground, $h = 0$.

$h = 30t - 5t^2 \quad \Rightarrow \quad 0 = 30t - 5t^2 \quad \Rightarrow \quad 5t^2 - 30\,t = 0$

$\Rightarrow\ 5t\,(t-6) = 0 \quad \Rightarrow \quad t = 0\ \ OR\ \ t = 6$ seconds Answer: $\boldsymbol{t} = \mathbf{6}$ seconds

(ii) Speed of ball at any time $= v = \frac{dh}{dt} = 30 - 10\,t$

∴ Speed of the ball after 2 seconds, v or $\frac{dh}{dt}$ at $t = 2$: $30 - 10(2) = 30 - 20 = \mathbf{10}$ m/s.

(iii) Height is maximum when v or $\frac{dh}{dt} = 0$, i.e. $30 - 10\,t = 0 \quad \Rightarrow \quad t = 3$ seconds

Maximum height $= h$ at $t = 3$: $h = 30t - 5t^2$

$= 30(3) - 5(3)^2 = 90 - 45 = \mathbf{45}$ m

(iv) Acceleration $= \frac{dv}{dt}$ or $\frac{d^2s}{dt^2} = \frac{d}{dt}(30 - 10\,t) = -10$ m/s^2

∴ Deceleration $= \mathbf{10}$ m/s^2.

(v) Speed of ball at any time $= v = \frac{dh}{dt} = 30 - 10\,t$

Speed of ball = 15 m/s when $= v = \frac{dh}{dt} = 30 - 10\,t = 15$

$\Rightarrow\ 15 = 10\,t \qquad \Rightarrow \qquad t = \mathbf{1.5}$ s

---

Example 31: A stone was dropped in a pond. The radius of a circle (or wave) produced increased at a rate of 75 cm/s. Find the rate at which the area inside the circle increased when the radius of the circle was 5 m.

Solution 31: **Given**: $\frac{dr}{dt} = 0.75$ [We cannot have a mixture of units, so bring lengths to metres]

Required to Find: $\frac{dA}{dt}$ @ $r = 5$

Express one variable in terms of the other: $A = \pi\, r^2$ [Formula for area of a disc/circle]

$\therefore \boldsymbol{\frac{dA}{dt}} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi r \times 0.75$ [Differentiating using chain rule and $\frac{d}{dr}(\pi\, r^2) = 2\pi r$]

$\therefore \boldsymbol{\frac{dA}{dt}}$ @ $r = 5$: $1.5\,\pi(5) = \frac{15\pi}{2}$ m^2/s.

Example 32: A large spherical balloon is being blown up and its radius is increasing at the rate of 5 cm/s. When the radius of the balloon is 30 cm, find the rate of increase of (i) its volume and (ii) its surface area.

Solution 32: **(i) Given**: $\frac{dr}{dt} = 5$ [All lengths are in cm, so there is no need to convert to metres]

Required to Find: $\frac{dV}{dt}$ @ $r = 30$

Express one variable in terms of the other: $V = \frac{4}{3}\pi r^3$ [Formula for volume of a sphere]

$\therefore \boldsymbol{\frac{dV}{dt}} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 5 = 20\pi r^2$. [Differentiating using chain rule and $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$]

$\therefore \boldsymbol{\frac{dV}{dt}}$ @ $r = 30$: $20\pi(30)^2 = 18\,000\,\pi$ cm³/s.

(ii) Given: $\frac{dr}{dt} = 5$ [All lengths are in cm, so there is no need to convert to metres]

Required to Find: $\frac{dA}{dt}$ @ $r = 30$

Express one variable in terms of the other: $A = 4\pi r^2$ [Formula for surface area of a sphere]

$\therefore \boldsymbol{\frac{dA}{dt}} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi r \times 5 = 40\pi r$. [Differentiating using chain rule and $\frac{d}{dr}(4\pi r^2) = 8\pi r$]

$\therefore \boldsymbol{\frac{dA}{dt}}$ @ $r = 30$: $40\pi(30) = 1200\,\pi$ cm²/s.

Maximum and Minimum Problems

The procedure for tackling maximum and minimum problems can be summarized as follows:

(1) Draw a diagram representing the problem (if appropriate) and label all the relevant details.

(2) Find a formula or equation for the quantity that is to be maximized or minimized.

(3) Express the quantity to be maximized (or minimized) in terms of one variable, using the conditions given in the problem, if necessary, to eliminate any other variable(s).

(4) Differentiate once to find where $\frac{dy}{dx} = 0$ (i.e. maximum or minimum point).

(5) Differentiate a second time to distinguish between the maximum and the minimum.

Example 33: A farmer wishes to enclose a rectangular section in the middle of a field. She uses all of $\mathbf{400}$ m of an electric fence to construct the enclosure.

(a) If one side of the rectangular enclosure is $\boldsymbol{x}$ metres long, what is the length of the adjacent side?

(b) Find, in terms of $\boldsymbol{x}$, the area of the enclosure.

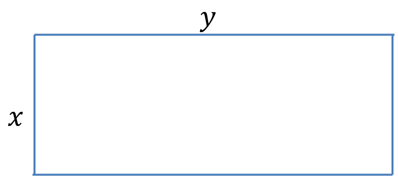
(c) Find the value of $\boldsymbol{x}$ that makes the area of the enclosure a maximum.

(d) If the farmer constructs a circular enclosure instead of a rectangular one (as above), find the radius of the enclosure correct to 2 decimal places, assuming she uses all $\mathbf{400}$ m of the electric fence to form the enclosure.

(e) Find the area of the circular enclosure to the nearest square metre.

(f) If the farmer wants the area of the enclosure to be a maximum (using exactly $\mathbf{400}$ m of electric fencing), should she construct a circular or a rectangular enclosure?

Solution 33: **(a)**



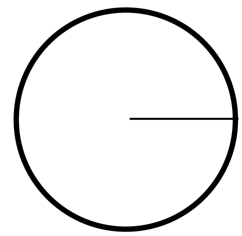
$$y = \frac{1}{2}(400 - 2x)$$
$$= 200 - x$$

[using info given to express one variable in terms of the other]

(b) To be maximized: Area(A) = length × breadth $= x(200 - x) = 200x - x^2$

(c) $\boldsymbol{\frac{dA}{dx} = 200 - 2x}$, which is a maximum when $\boldsymbol{\frac{dA}{dx} = 200 - 2x = 0 \Rightarrow x = 100.}$

$\frac{d^2A}{dx^2} = -2 < 0 \Rightarrow$ Maximum @ $x = 100$



(d) circumference length$= 2\pi r \Rightarrow 400 = 2\pi r$

$\Rightarrow r = \frac{400}{2\pi} = \frac{200}{\pi} = 63.66$ m

(e) Area $= \pi r^2 = \pi\,(63.66)^2 = 12\,732$ m^2

(f) Maximum rectangular area $= 100 \times 100 = 10\,000$ m^2

⇒ biggest area formed by circle

Example 34: If two natural numbers add to give **30**, find the maximum product of these two numbers.

Solution 34: Let the two numbers be x and y. But, $y = 30 - x$ [using info given to express one variable in terms of the other]

To be maximized: Product $(P) = x \times (30 - x) = 30x - x^2$

$\frac{dP}{dx} = 30 - 2x$, which is a maximum when $\frac{dP}{dx} = 30 - 2x = 0 \Rightarrow x = 15.$

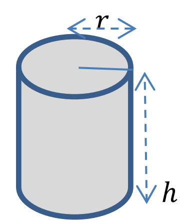
$\frac{d^2A}{dx^2} = -2 < 0 \Rightarrow$ Maximum @ $x = 15$

Maximum product $xy = 15(30 - 15) = 225$

Example 35: A cylindrical can is used to hold **330** cm^3 of orange drink.

(i) Find the height (h) of the can in terms of the radius (r).

(ii) Find the dimensions of the can that will use the least amount of metal in its manufacture.



Solution 35: **(i)** Volume of cylinder$= 330 = \pi r^2 h \Rightarrow h = \frac{330}{\pi r^2}$

(ii) To be minimized: Surface Area $(A) = 2\pi r h + 2(\pi r^2)$

$$= 2\pi r \frac{330}{\pi r^2} + 2\pi r^2$$
$$= \frac{660}{r} + 2\pi r^2 = 660 r^{-1} + 2\pi r^2$$

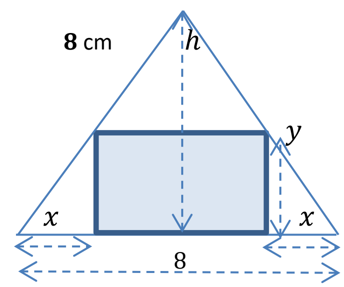
$\therefore \frac{dA}{dr} = -660\, r^{-2} + 4\pi r$, which is a maximum when $\frac{dA}{dr} = -660\, r^{-2} + 4\pi r = 0$

$$\Rightarrow \frac{660}{r^2} = 4\pi r$$
$$\Rightarrow 660 = 4\pi r^3$$
$$\Rightarrow r = \sqrt[3]{\frac{660}{4\pi}} = \sqrt[3]{\frac{165}{\pi}} = 3.745 \text{ cm}$$

Note: $\frac{d^2A}{dr^2} = \frac{1320}{r^3} + 4\pi$ which is > 0 when $r = 3.745 \Rightarrow$ Minimum @ $r = 3.745$.

When $r = 3.745$: $h = \frac{330}{\pi r^2} = \frac{330}{\pi(3.745)^2} = \frac{330}{44.06} = 7.49$ cm

Example 36: An equilateral triangle of length $\mathbf{8}$ cm has a rectangle inscribed in it.



(a) Find h, the height of the triangle.

(b) Show that $\boldsymbol{y = \sqrt{3}\ x}$.

(c) Find, in terms of $\boldsymbol{x}$ the area of the rectangle.

(d) Hence, find the maximum possible area of this rectangle.

(e) What percentage of the triangle is occupied by the maximum sized rectangle?

Solution 36: **(a)** By Pythagoras' theorem: $h^2 = 8^2 - 4^2 = 64 - 16 = 48$

$$\Rightarrow \quad h = \sqrt{48} \ or \ 4\sqrt{3} \text{ cm}$$

(b) $tan\ 60^o = \frac{y}{x} \quad \Rightarrow \quad \sqrt{3} = \frac{y}{x} \quad \Rightarrow \quad y = \sqrt{3}\, x$

(c) Area of Rectangle $(A) = (8 - 2x)\,(y)$

$$= (8 - 2x)\left(\sqrt{3}\, x\right)$$

$$= 8\sqrt{3}\, x - 2\sqrt{3}\, x^2 \text{ cm}^2$$

(d) $\frac{dA}{dx} = 8\sqrt{3} - 4\sqrt{3}\, x,$

$\therefore$ Area is a maximum when $\frac{dA}{dx} = 8\sqrt{3} - 4\sqrt{3}\, x = \mathbf{0}$

$$\Rightarrow \quad x = \frac{8\sqrt{3}}{4\sqrt{3}} = 2$$

But, $\frac{d^2A}{dx^2} = -4\sqrt{3} < 0 \quad \Rightarrow$ Maximum @ $x = 2$

$$\therefore \quad A_{max} = 8\sqrt{3}\,(2) - 2\sqrt{3}\,(2)^2$$

$$= 16\sqrt{3} - 8\sqrt{3}$$

$$= 8\sqrt{3} \ \textbf{cm}^2$$

(e) Area of Triangle $= \frac{1}{2}\,(8)\left(4\sqrt{3}\right)$

$$= 16\sqrt{3} \text{ cm}^2$$

$\therefore$ % Area occupied by maximum rectangle $= \frac{8\sqrt{3}}{16\sqrt{3}} \times \frac{100}{1}$

$$= 50\%$$

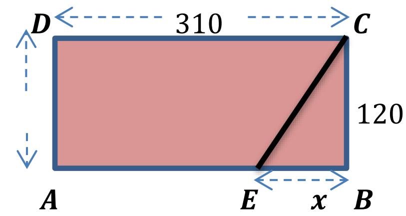
Example 37: $\boldsymbol{ABCD}$ is a rectangular ploughed field $\mathbf{310}$ m long and $\mathbf{120}$ m wide with a path around its sides.

Tom can run at a speed of $^{13}/_{3}$ m/s along the path

and at a speed of $^{5}/_{3}$ m/s across the ploughed field.

Tom wants to reach point $\boldsymbol{C}$ from the point $\boldsymbol{A}$ as quickly as possible.

He runs from $\boldsymbol{A}$ to a point $\boldsymbol{E}$ along the path and from $\boldsymbol{E}$ to $\boldsymbol{C}$ across the field.



(a) If $|\boldsymbol{EB}| = \boldsymbol{x}$, express in terms of $\boldsymbol{x}$:

(i) $|\boldsymbol{AE}|$

(ii) $|\boldsymbol{EC}|$

(iii) the total time taken to travel from $\boldsymbol{A}$ to $\boldsymbol{C}$

(b) Find the value of $\boldsymbol{x}$ for which the time is a minimum.

(c) Calculate the minimum time to travel from $\boldsymbol{A}$ to $\boldsymbol{C}$ (in seconds).

Solution 37: **(a)** **(i)** $|\boldsymbol{AE}| = 310 - x$

(ii) $|\boldsymbol{EC}|^2 = x^2 + 120^2$

$\Rightarrow |\boldsymbol{EC}| = \sqrt{x^2 + 120^2}$

(iii) time taken $= \dfrac{distance}{speed}$

$$= \frac{(3)\,(310 - x)}{13} + \frac{(3)\sqrt{x^2 + 120^2}}{5}$$

(b) **To be minimized: time** $= t = \dfrac{(3)\,(310-x)}{13} + \dfrac{(3)\sqrt{x^2+120^2}}{5}$

$\Rightarrow \dfrac{dt}{dx} = \dfrac{-3}{13} + \dfrac{3}{5}\dfrac{1}{2}(x^2 + 120^2)^{\frac{-1}{2}}(2x) = \mathbf{0}$ [when time is a minimum]

$\Rightarrow \dfrac{3}{13} = \dfrac{3}{5}\dfrac{x}{\sqrt{x^2+120^2}}$

$\Rightarrow \mathbf{13}x = \mathbf{5}\sqrt{x^2 + 120^2}$

$\Rightarrow \mathbf{169}x^2 = \mathbf{25}\,(x^2 + 120^2)$

$\Rightarrow \mathbf{144}x^2 = \mathbf{25}\,(120^2)$

$\Rightarrow \mathbf{12}x = \pm\mathbf{5}\,(120)$

$\Rightarrow x = \pm\mathbf{5}\,(10) = \pm\,\mathbf{50}$

$\Rightarrow x = \mathbf{50}$

$\therefore$ **time is a minimum when** $\boldsymbol{x} = \mathbf{50}$ **m**

(c) **Minimum time** $= \dfrac{(3)\,(310-x)}{13} + \dfrac{(3)\sqrt{x^2+120^2}}{5}$

$= \dfrac{(3)\,(310-50)}{13} + \dfrac{(3)\sqrt{50^2+120^2}}{5}$

$= 60 + \dfrac{(3)130}{5}$

$= 60 + 78$

$= 138$ s

Sketching Derivative Curves

If we are given the graph of a curve $y = f(x)$, we can sketch the graph of its derivative, $\frac{dy}{dx}$ $or\ f'(x)$, by examining the sign of the derivative and how the slope of the tangent is changing.

The diagram below shows a cubic function $y = f(x)$, with turning points at $(0,\ 8)$ and $(1.1,\ 6)$.
Also shown are the functions $f'(x)$ and $f''(x)$.

***Note**:* 1) At a maximum or minimum of one curve, the derivative curve will cut the x-axis

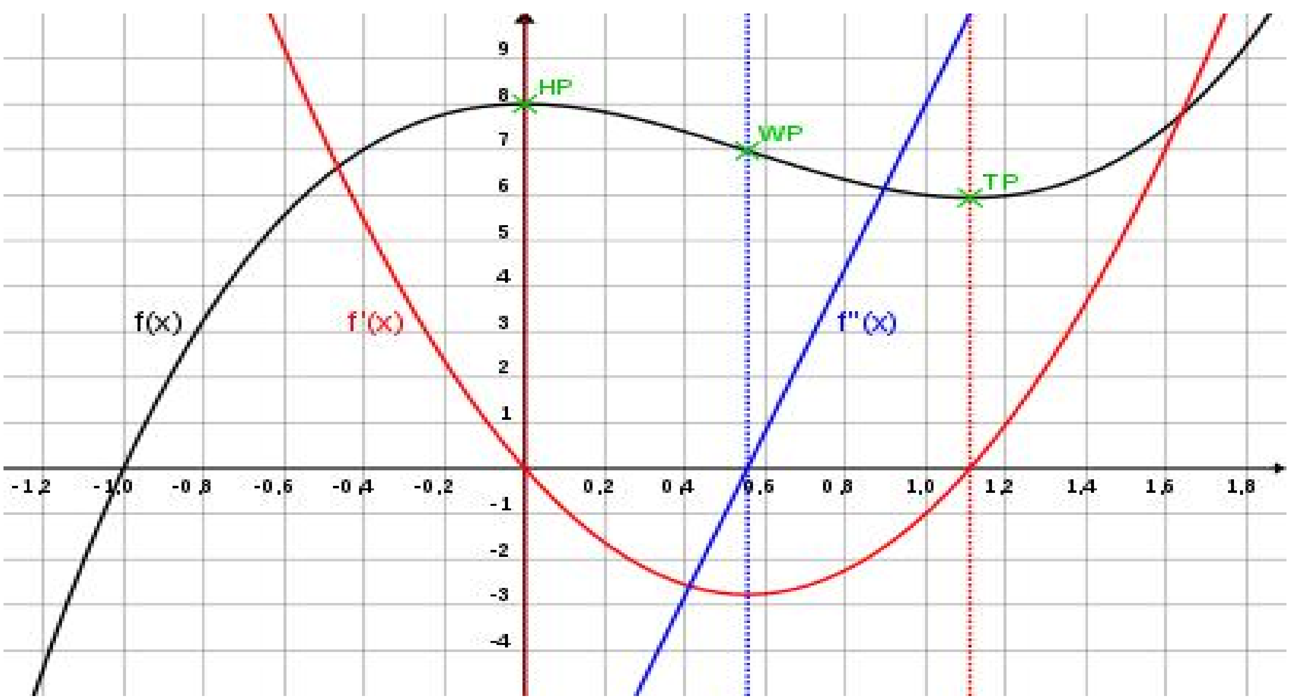
i.e. $\frac{dy}{dx} = 0$ at a maximum and at a minimum

2) At a point of inflection of one curve, the derivative curve will have a turning point and the second derivative curve will cut the x −axis.

i.e. $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at a point of inflection

3) After that, check whether the slope of the original is positive or negative and whether it is increasing or decreasing.

Minimum at $(1.1, 6)$ $\Rightarrow f'(1.1) = 0$ $\Rightarrow$ $(1.1, 0)$ is on the derivative curve $f'(x)$.
Maximum at $(0, 8)$ $\Rightarrow f'(0) = 0$ $\Rightarrow$ $(0, 0)$ is on the derivative curve $f'(x)$.
A point of inflection at $x = 0.55$ on $y = f(x)$ $\Rightarrow$ turning point at $x = 0.55$ on $f'(x)$
AND $(0.55, 0)$ is on the second derivative curve $f''(x)$
Maximum at $x = 0$ on $y = f(x)$ $\Rightarrow$ $(0, 0)$ is on the derivative curve, $y = f'(x)$.
Slope of $y = f'(x)$ is negative but decreasing negatively to the left of $x = 0.55$, but is positive and increasing to the right of $x = 0.55$ $\Rightarrow$ $y = f''(x)$ has a positive slope, cutting x-axis at $(0.55,\ 0)$.



Implicit Differentiation (2015+)

[Syllabus Addition (2015): Use differentiation to find the slope of a tangent to a circle**]**

It is easy to find $\frac{d\,(x^2)}{dx}$. The answer is $2x$. But, what if we are asked to find $\frac{d\,(y^2)}{dx}$.

The technique for doing this is known as **implicit differentiation**. We have to use the **Chain Rule** to help us out.

$\frac{d\,(y^2)}{dx} = \frac{d\,y^2}{dy}\,\frac{dy}{dx}$ *(i.e. multiply above and below by dy) (not forgetting the sideways kick).* The answer is: $2y\,\frac{dy}{dx}$.

Examples: (i) $\frac{d\,(7y^3)}{dx} = \frac{d\,(7y^3)}{dy}\,\frac{dy}{dx} = 21y^2\,\frac{dy}{dx}$

(ii) $\frac{d\,(\sqrt{y})}{dx} = \frac{d\,(\sqrt{y})}{dy}\,\frac{dy}{dx} = \frac{1}{2}y^{-1/2}\,\frac{dy}{dx} = \frac{1}{2\sqrt{y}}\,\frac{dy}{dx}$

Example 38: A circle has an equation $x^2 + y^2 - 4x + 2y - 20 = 0$.
Using differentiation, find the equation of the tangent to this circle at the point $(5, -5)$.

Solution38:

Step 1: Differentiate both sides of the equation with respect to x.

$$2x + 2y\,\frac{dy}{dx} - 4 + 2\,\frac{dy}{dx} - 0 = 0$$

Step 2: Gather up all the terms with $\frac{dy}{dx}$ on one side of the equation and isolate $\frac{dy}{dx}$.

$$2y\,\frac{dy}{dx} + 2\,\frac{dy}{dx} = 4 - 2x$$
$$\Rightarrow\ (2y+2)\,\frac{dy}{dx} = 4 - 2x$$
$$\Rightarrow\ \frac{dy}{dx} = \frac{4 - 2x}{2y + 2}$$

Step 3: Find the slope of the tangent at the given point [find $\frac{dy}{dx}$ at $(x,\ y) = (5, -5)$]

$\frac{dy}{dx}$ at $(x,\ y) = (5, -5)$ is: $\frac{4 - 2(5)}{2(-5) + 2} = \frac{-6}{-8} = \frac{3}{4}$

Step 4: Find the equation of the tangent, given a point on it $(x_1\,,\ y_1)$ and its slope (m):

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow\ y - (-5) = \tfrac{3}{4}(x - 5)$$
$$\Rightarrow\ y + 5 = \tfrac{3}{4}(x - 5)$$
$$\Rightarrow\ 4y + 20 = 3x - 15$$
$$\Rightarrow\ 3x - 4y - 35 = 0$$

POINTS TO REMEMBER

1) If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$. $\frac{dy}{dx}$ or $f'(x)$ has many names. It can be called the **differential of y with respect to x**, the **derivative of y or $f(x)$**, the **derived function** of $f(x)$, the differential coefficient of $f(x)$, **the slope of the tangent to the curve** at (x, y) or $(x, f(x))$, the **gradient**. [***Note***: $f'(2)$ is the value of $f'(x)$ when $x = 2$]

2) $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

3) $\lim\limits_{x\to a}[f(x).\ g(x)] = \lim\limits_{x\to a} f(x)\ .\ \lim\limits_{x\to a} g(x)$

4) $\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

5) $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

6) $\frac{dy}{dx} = \lim\limits_{h\to 0}\frac{f(x+h) - f(x)}{h}$ where $y = f(x)$.

7) If $y = f(x) = x^n$, then $\frac{dy}{dx}$ or $f'(x) = n\,x^{n-1}$. (*)

8) $y = k\,f(x)$, $\frac{dy}{dx} = k\,f'(x)$, where k is a constant. [Multiplying constants can be brought outside and left there.]

Example: If $y = 3x^5$, $\frac{dy}{dx} = 3\,f'(x^5) = 3\,(5x^4) = 15x^4$

9) If $f(x) = u(x) + v(x)$, then $\frac{df}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Example: If $y = 3x^9 + 5x^2$, $\frac{dy}{dx} = 27x^8 + 10x$.

10) If $f(x) = u(x)v(x)$, then $\frac{d\,f}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. (*)

Example: If $y = (5x+2)x^3$, $\frac{dy}{dx} = (5x+2)3x^2 + (x^3)(5) \Rightarrow \frac{dy}{dx} = 20x^3 + 6x^2$.

11) If $f(x) = \frac{u(x)}{v(x)}$, then $\boxed{\frac{df}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$. (*)

Example: If $f(x) = \frac{x^2 - 5}{7x}$, $\frac{dy}{dx} = \frac{(7x)(2x) - (x^2-5)(7)}{(7x)^2} \Rightarrow \frac{dy}{dx} = \frac{7x^2 + 35}{49x^2}$

12) If $y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n\,[f(x)]^{n-1}\,[f'(x)]$.

Example: If $y = (4x^3 - 5x + 1)^7$, $\frac{dy}{dx} = 7(4x^3 - 5x + 1)^6\,(12x^2 - 5) = (84x^2 - 35)\,(4x^3 - 5x + 1)^6$

13) If $y = sin\,x$, $\frac{dy}{dx} = cos\,x$. If $y = sin\left(f(x)\right)$, $\frac{dy}{dx} = [cos(f(x)]\,[f'(x)]$.

Example: If $y = sin\,14x^2$, $\frac{dy}{dx} = (cos\,14x^2)\,(28x)$.

14) If $y = cos\,x$, $\frac{dy}{dx} = -sin\,x$. If $y = cos\left(f(x)\right)$, $\frac{dy}{dx} = [-sin(f(x)]\,[f'(x)]$.

Example: If $y = cos\,5x$, $\frac{dy}{dx} = (-sin\,5x)\,(5)$ or $-5\,sin\,5x$.

15) If $y = tan\,x$, $\frac{dy}{dx} = sec^2\,x$. If $y = tan\left(f(x)\right)$, $\frac{dy}{dx} = [sec^2(f(x)]\,[f'(x)]$.

Example: If $y = tan\,8x$, $\frac{dy}{dx} = (sec^2\,8x)\,(8)$ or $8\,sec^2\,8x$.

16) If $y = sin^{-1}x$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$. *If* $y = sin^{-1}f(x)$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-[f(x)]^2}} \times f'(x)$.

Example: If $y = sin^{-1}\sqrt{x}$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2}\,(x)^{\frac{-1}{2}} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\,\sqrt{1-x}} = \frac{1}{2\,\sqrt{x-x^2}}$

17) If $y = sin^{-1}\frac{x}{a}$, $\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$.

Example: If $y = sin^{-1}\frac{5x^3}{4}$, $\frac{dy}{dx} = \frac{1}{\sqrt{16 - (5x^3)^2}} \times 15x^2$

18) If $y = tan^{-1}x, \frac{dy}{dx} = \frac{1}{1+x^2}$. *If* $y = tan^{-1}f(x)$, $\frac{dy}{dx} = \frac{1}{1+[f(x)]^2} \times f'(x)$.

Example: If $y = tan^{-1}(7x-3)$, $\frac{dy}{dx} = \frac{1}{1+(7x-3)^2} \times 7 = \frac{7}{1+(7x-3)^2} = \frac{7}{49x^2 - 42x + 10}$

19) If $y = tan^{-1}\frac{x}{a}$, $\frac{dy}{dx} = \frac{a}{a^2 + x^2}$.

Example: If $y = tan^{-1}\frac{5x^3}{4}$, $\frac{dy}{dx} = \frac{4}{(16+(5x^3)^2} \times 15x^2$ or $\frac{60x^2}{16+25x^6}$.

20) If $y = e^x, \frac{dy}{dx} = e^x$. If $y = e^{f(x)}, \frac{dy}{dx} = e^{f(x)} \times f'(x)$.

Example: If $y = e^{-9x+5}, \frac{dy}{dx} = e^{-9x+5} \times -9 = -9\,e^{-9x+5}$

21) If $= log_e x, \frac{dy}{dx} = \frac{1}{x}$. If $y = log_e f(x), \frac{dy}{dx} = \frac{f'(x)}{f(x)}$. [i.e. the differential of the function over the function]

Example: If $y = log_e(2x^5 - 6x + 7)$, $\frac{dy}{dx} = \frac{10x^4 - 6}{2x^5 - 6x + 7}$.

22) If $y = f(x)$, then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$.

Example: If $f(x) = x^2 - 4x + 7 \quad \Rightarrow \quad f'(x) = 2x - 4 \quad \Rightarrow \quad f''(x) = 2.$

23) Before attempting to differentiate difficult log functions, check if the question be made a easier by applying the rules of logs.

24) $\frac{dy}{dx}$ is the **slope of a tangent** to a curve at any point on the curve.

25) To find a **turning point** on a graph , use $\frac{dy}{dx} = 0$ and solve the resulting equation.

26) At ***local maximum***: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

27) At ***local minimum***: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

28) At ***point of inflection***: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$

29) Curves are ***increasing*** when $\frac{dy}{dx} > 0$. Curves are ***decreasing*** when $\frac{dy}{dx} < 0$.

30) In reciprocal graphs such as $\boldsymbol{y = \frac{px+q}{cx+d}}$, the vertical asymptote is got by letting the denominator $= 0 \Rightarrow \boldsymbol{x = \frac{-d}{c}}$.

Horizontal asymptotes are got by letting $y = \lim_{x\to\infty} f(x)$ and dividing above and below by highest power of x.

Here, horizontal asymptote is: $= \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{p+\frac{q}{x}}{c+\frac{d}{x}} = \boldsymbol{\frac{p+0}{c+0}} \Rightarrow \boldsymbol{y = \frac{p}{c}}$.

31) If s = distance travelled or displacement, $v = \frac{ds}{dt}$ = speed or velocity and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ = acceleration.

32) With questions on rates of change of one variable with respect to another, always write down the "gimme", (e.g. $\frac{dr}{dt} = 0.75$) , then write down what you are trying to find (e.g. $\frac{dA}{dt}$ @ $r = 5$), do the sideway kick (e.g. $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$, then express one variable in terms of the other (e.g. $A = \pi r^2$) and differentiate (e.g. $\frac{dA}{dr} = 2\pi r$).

Finally, put in the "at" value given.

33) Familiarise yourself with formulae available on pages **25** and **27** of Mathematical Formulae and Tables.