Trig – Identities Solutions

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Exercise 1

1. Express each of the following in surd form, without using a calculator.

(a) $sin\,75° = sin(30 + 45)$

$sin(A + B) = sin\,A\,cos\,B + cos\,A\,sin\,B$

$sin(30 + 45) = sin\,30\,cos\,45 + cos\,30\,sin\,45$

$$sin\,75° = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

(b) $cos\,105° = cos(60 + 45)$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(60 + 45) = \cos 60\cos 45 - \sin 60\sin 45$

$$\cos 105° = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

(c) $tan\,75° = tan(45 + 30)$

$$tan(A + B) = \frac{tan\,A + tan\,B}{1 - tan\,A\,tan\,B}$$

$$tan(45 + 30) = \frac{tan\,45 + tan\,30}{1 - tan\,45\,tan\,30}$$

$$tan\,15° = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \left(\frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\left(\frac{1}{\sqrt{3}}\right)}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= 2 + \sqrt{3}$$

(d) $cos\,15° = cos(60 - 45)$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\cos(60 - 45) = \cos 60\cos 45 + \sin 60\sin 45$

$$\cos 15° = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

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(e) $tan\,105°$

$$tan(A+B) = \frac{tan\,A + tan\,B}{1 - tan\,A\,tan\,B}$$

$$tan(60+45) = \frac{tan\,60 + tan\,45}{1 - tan\,60\,tan\,45}$$

$$tan\,105° = \frac{\sqrt{3}+1}{1-\sqrt{3}(1)}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

$$= -2-\sqrt{3}$$

(f) $sin\,165°$

$$sin(A-B) = sin\,A\,cos\,B - cos\,A\,sin\,B$$

$$sin(180-15) = sin\,180\,cos\,15 - cos\,180\,sin\,15$$

$$sin\,165° = (0)(cos\,15) - (-1)(sin\,15)$$

$$= \sin 15$$

$$\sin 15 = \sin(45-30)$$

$$sin(A-B) = sin\,A\,cos\,B - cos\,A\,sin\,B$$

$$sin(45-30) = sin\,45\,cos\,30 - cos\,45\,sin\,30$$

$$sin\,15° = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

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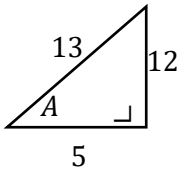
2. A and B are acute angles. $cos\ A = \frac{5}{13}$ and $sin\ B = \frac{3}{5}$. Work out the value of each of the following, without using a calculator.

We know $\cos A = \frac{5}{13}$, we need $\sin A$ and $\tan A$.

By Pythagoras:

$$\cos A = \frac{5}{13} = \frac{a}{h}$$

$$h^2 = o^2 + a^2$$
$$13^2 = o^2 + 5^2$$
$$\sqrt{169 - 25} = o$$
$$12 = o$$

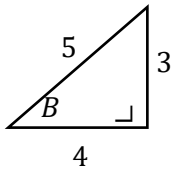


We know $\sin B = \frac{3}{5}$, we need $\cos B$ and $\tan B$.

By Pythagoras:

$$\sin B = \frac{3}{5} = \frac{o}{h}$$

$$h^2 = o^2 + a^2$$
$$5^2 = 3^2 + a^2$$
$$\sqrt{25 - 9} = a$$
$$4 = a$$



$\sin A = \frac{o}{h}$ $\sin A = \frac{12}{13}$	$\tan A = \frac{o}{a}$ $\tan A = \frac{12}{5}$	$\cos B = \frac{a}{h}$ $\cos B = \frac{4}{5}$	$\tan B = \frac{o}{a}$ $\tan B = \frac{3}{4}$

(a) $\sin A = \frac{12}{13}$

(b) $\tan A = \frac{12}{5}$

(c) $\cos B = \frac{4}{5}$

(d) $\tan B = \frac{3}{4}$

(e) $sin(A - B)$
$= sin\ A\ cos\ B - cos\ A\ sin\ B$
$= \left(\frac{12}{13}\right)\left(\frac{4}{5}\right) - \left(\frac{5}{13}\right)\left(\frac{3}{5}\right)$
$= \frac{33}{65}$

(f) $cos(A - B)$
$= \cos A \cos B + \sin A \sin B$
$= \left(\frac{5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)$
$= \frac{4}{13} - \frac{9}{25}$
$= \frac{56}{65}$

(g) $tan(A + B)$
$= \frac{tan\ A + tan\ B}{1 - tan\ A\ tan\ B}$
$= \frac{\frac{12}{5} + \frac{3}{4}}{1 - \left(\frac{12}{5}\right)\left(\frac{3}{4}\right)}$
$= -\frac{63}{16}$

(h) $\sin 2A = 2 \sin A \cos A$
$= 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right)$
$= \frac{120}{169}$

(i) $tan\ 2A = \frac{2 \tan A}{1 - \tan^2 A}$
$= \frac{2\left(\frac{12}{5}\right)}{1 - \left(\frac{12}{5}\right)^2}$
$= -\frac{120}{119}$

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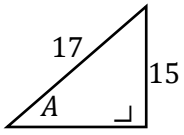
3. A and B are acute angles. $sin\ A = \frac{15}{17}$ and $cos\ B = \frac{3}{\sqrt{13}}$. Work out the value of each of the following, without using a calculator. Where appropriate, give your answer in surd form.

We know $\sin A = \frac{5}{17}$, we need $\cos A$ and $\tan A$.

By Pythagoras:

$$\sin A = \frac{15}{17} = \frac{o}{h}$$

$$h^2 = o^2 + a^2$$
$$17^2 = 15^2 + a^2$$
$$\sqrt{17^2 - 15^2} = a$$
$$8 = a$$

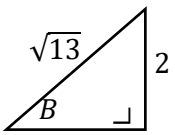


We know $\cos B = \frac{3}{\sqrt{13}}$, we need $\sin B$ and $\tan B$.

By Pythagoras:

$$\cos B = \frac{3}{\sqrt{13}} = \frac{a}{h}$$

$$h^2 = o^2 + a^2$$
$$(\sqrt{13})^2 = o^2 + 3^2$$
$$\sqrt{13 - 9} = o$$
$$2 = o$$



$$\cos A = \frac{a}{h}$$
$$\cos A = \frac{8}{17}$$

$$\tan A = \frac{o}{a}$$
$$\tan A = \frac{15}{8}$$

$$\sin B = \frac{o}{h}$$
$$\sin B = \frac{2}{\sqrt{13}}$$

$$\tan B = \frac{o}{a}$$
$$\tan B = \frac{2}{3}$$

(a) $\sin(A + B)$

$$= \sin A \cos B + \cos A \sin B$$
$$= \left(\frac{15}{17}\right)\left(\frac{3}{\sqrt{13}}\right) + \left(\frac{8}{17}\right)\left(\frac{2}{\sqrt{13}}\right)$$
$$= \frac{61}{17\sqrt{13}}$$
$$= \frac{61\sqrt{13}}{221}$$

(b) $\cos 2A = \cos^2 A - \sin^2 A$

$$= \left(\frac{8}{17}\right)^2 - \left(\frac{15}{17}\right)^2$$
$$= -\frac{169}{289}$$

(c) $\tan(A - B)$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$
$$= \frac{\frac{15}{8} - \frac{2}{3}}{1 + \left(\frac{15}{8}\right)\left(\frac{2}{3}\right)}$$
$$= \frac{29}{54}$$

4. $tan\ X = \frac{4}{3}$, where $0° < X < 90°$. Evaluate, without evaluating X.

(a) $\sin 2X = \frac{2\tan X}{1 + \tan^2 X}$

$$= \frac{2\left(\frac{4}{3}\right)}{1 + \left(\frac{4}{3}\right)^2}$$
$$= \frac{24}{25}$$

(b) $\cos 2X = \frac{1 - \tan^2 X}{1 + \tan^2 X}$

$$= \frac{1 - \left(\frac{4}{3}\right)^2}{1 + \left(\frac{4}{3}\right)^2}$$
$$= -\frac{7}{25}$$

(c) $\tan 2X = \frac{2\tan X}{1 - \tan^2 X}$

$$= \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2}$$
$$= -\frac{24}{7}$$

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5. Given that $tan\ \theta = \frac{1}{3}$, show that $tan\ 2\theta = \frac{3}{4}$.

(d) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$= \dfrac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}$

$= \frac{3}{4}$

6. If $cos\ \theta = \frac{1}{\sqrt{3}}$, work out the value of $cos\ 2\theta$, without evaluating θ.

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2$

$= -\frac{1}{3}$

We know $\cos\theta = \frac{1}{\sqrt{3}}$, we need $\sin\theta$.

By Pythagoras:

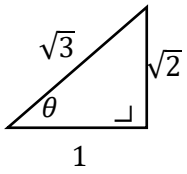
$$\cos\theta = \frac{1}{\sqrt{3}} = \frac{a}{h}$$

$h^2 = o^2 + a^2$

$(\sqrt{3})^2 = o^2 + 1^2$

$\sqrt{3-1} = o$

$\sqrt{2} = o$



$\sin\theta = \frac{o}{h}$

$\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$

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7. $tan\,A = 4$ and $tan(A + B) = 5$, where $0° < A < 90°$ and $0° < B < 90°$.
Work out the value of $tan\,B$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$5 = \frac{4 + \tan B}{1 - 4\tan B}$$

Let $\tan B = x$:

$$5 = \frac{4 + x}{1 - 4x}$$

$$5(1 - 4x) = 4 + x$$

$$5 - 20x = 4 + x$$

$$1 = 21x$$

$$\frac{1}{21} = x$$

Swap back: $\tan B = \frac{1}{21}$

8. $tan\,A = 2$ and $tan(A + B) = 3$, where $0° < A < \frac{\pi}{2}$ and $0° < B < \frac{\pi}{2}$.
Work out the value of $tan\,B$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$3 = \frac{2 + \tan B}{1 - 2\tan B}$$

Let $\tan B = x$:

$$3 = \frac{2 + x}{1 - 2x}$$

$$3(1 - 2x) = 2 + x$$

$$3 - 6x = 2 + x$$

$$1 = 7x$$

$$\frac{1}{7} = x$$

Swap back: $\tan B = \frac{1}{7}$

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9. A and B are acute angles such that $tan\, B = \frac{1}{3}$ and $tan(A - B) = 2$. Without evaluating A or B, work out the value of: **(a)** $tan\, A$ **(b)** $cos\, 2A$.

(a)

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$2 = \frac{\tan A - \frac{1}{3}}{1 + \tan A\left(\frac{1}{3}\right)}$$

Let $\tan A = x$:

$$2 = \frac{x - \frac{1}{3}}{1 + x\left(\frac{1}{3}\right)}\left(\frac{3}{3}\right)$$

$$2 = \frac{3x - 1}{3 + x}$$

$$2(3 + x) = 3x - 1$$

$$6 + 2x = 3x - 1$$

$$7 = x$$

Swap back: $\tan A = 7$

(b)

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$= \frac{1 - (7)^2}{1 + (7)^2}$$

$$= -\frac{24}{25}$$

10. If $sin\, 2A = \frac{15}{17}$ and $0° < A < 90°$. Find two possible values of $tan\, A$, without evaluating A.

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\frac{15}{17} = \frac{2 \tan A}{1 + \tan^2 A}$$

Let $x = \tan A$:

$$\frac{15}{17} = \frac{2x}{1 + x^2}$$

$$15(1 + x^2) = 17(2x)$$

$$15 + 15x^2 = 34x$$

$$15x^2 - 34x + 15 = 0$$

$$(3x - 5)(5x - 3) = 0$$

$$x = \frac{5}{3}, \quad x = \frac{3}{5}$$

Swap back:

$$\tan A = \frac{5}{3}, \tan A = \frac{3}{5}$$

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11. If $sin\ 2X = \frac{24}{25}$ and $0° < X < 90°$. Find two possible values of $tan\ X$, without evaluating X.

$\sin 2X = \frac{2\tan X}{1+\tan^2 X}$

$\frac{24}{25} = \frac{2\tan X}{1+\tan^2 X}$

Let $x = \tan X$:

$$\frac{24}{25} = \frac{2x}{1+x^2}$$

$$24(1+x^2) = 25(2x)$$

$$24 + 24x^2 = 50x$$

$$24x^2 - 50x + 24 = 0$$

$$12x^2 - 25x + 12 = 0$$

$$(3x-4)(4x-3) = 0$$

$x = \frac{4}{3}$, $x = \frac{3}{4}$

Swap back:

$\tan X = \frac{4}{3}, \tan X = \frac{3}{4}$

12. If $cos\ 2B = \frac{12}{13}$ and $0° < B < \frac{\pi}{2}$. Find the value of $cos\ B$, without evaluating B.

$\cos^2 B = \frac{1}{2}(1+\cos 2B)$

$\cos^2 B = \frac{1}{2}\left(1+\frac{12}{13}\right)$

$\cos^2 B = \frac{25}{26}$

$\cos B = \pm\sqrt{\frac{25}{26}}$

$\cos B = \pm\frac{5}{\sqrt{26}}$ But $0° < B < \frac{\pi}{2}$ and cosine is positive in quadrant 1 (acute angles), so:

$\cos B = \frac{5}{\sqrt{26}}$

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13. If $cos\ 2\theta = \frac{7}{25}$ and $0° < \theta < \frac{\pi}{2}$. Find the value of $cos\ \theta$, without evaluating θ.

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\cos^2\theta = \frac{1}{2}\left(1 + \frac{7}{25}\right)$

$\cos^2\theta = \frac{16}{25}$

$\cos\theta = \pm\sqrt{\frac{16}{25}}$

$\cos\theta = \pm\frac{4}{5}$ But $0° < \theta < \frac{\pi}{2}$ and cosine is positive in quadrant 1 (acute angles), so:

$\cos\theta = \frac{4}{5}$

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Exercise 2

Write each of the following as a sum or difference of two trigonometric functions.

1. $2\sin 5x\cos 3x$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\sin 5x\cos 3x = \sin(5x+3x) + \sin(5x-3x)$$
$$= \sin 8x + \sin 2x$$

2. $2\cos 7y\cos y$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\cos 7y\cos y = \cos(7y+y) + \cos(7y-y)$$
$$= \cos 8y + \cos 6y$$

3. $2\cos 6\theta\cos 2\theta$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\cos 6\theta\cos 2\theta = \cos(6\theta+2\theta) + \cos(6\theta-2\theta)$$
$$= \cos 8\theta + \cos 4\theta$$

4. $-2\sin 5A\sin A$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$-2\sin 5A\sin A = -[\cos(5A-A) - \cos(5A+A)]$$
$$= -[\cos 4A - \cos 6A]$$
$$= \cos 6A - \cos 4A$$

5. $\cos x\sin 6x$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$\cos x\sin 6x = \frac{1}{2}[\sin(x+6x) - \sin(x-6x)]$$
$$= \frac{1}{2}[\sin 7x - \sin(-5x)]$$
$$= \frac{1}{2}[\sin 7x + \sin 5x]$$

$\sin(-A) = -\sin A$

Formulae and Tables Book, p.13

6. $\cos 5B\sin B$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$\cos 5B\sin B = \frac{1}{2}[\sin(5B+B) - \sin(5B-B)]$$
$$= \frac{1}{2}[\sin 6B - \sin 4B]$$

Write each of the following as a product of two trigonometric functions.

7. $\sin 6x + \sin 2x$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin 6x + \sin 2x = 2\sin\frac{6x+2x}{2}\cos\frac{6x-2x}{2}$$
$$= 2\sin 4x\cos 2x$$

8. $\cos 10\theta - \cos 4\theta$

$$\cos A + \cos B = -2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos 10\theta + \cos 4\theta = -2\cos\frac{10\theta+4\theta}{2}\cos\frac{10\theta-4\theta}{2}$$
$$= -2\cos 7\theta\cos 3\theta$$

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9. $sin\,7y - sin\,3y$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\sin 7y - \sin 3y = 2\cos\frac{7y+3y}{2}\sin\frac{7y-3y}{2}$$
$$= 2\cos 5y \sin 2y$$

10. $cos\,2x + cos\,6x$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos 2x + \cos 6x = 2\cos\frac{2x+6x}{2}\cos\frac{2x-6x}{2}$$
$$= 2\cos 4x \cos(-2x)$$
$$= 2\cos 4x \cos 2x$$

$\boldsymbol{\cos(-A) = \cos A}$

Formulae and Tables Book, p.13

11. $sin\,4A - sin\,12A$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\sin 4A - \sin 12A = 2\cos\frac{4A+12A}{2}\sin\frac{4A-12A}{2}$$
$$= 2\cos 8A \sin(-4A)$$
$$= -2\cos 8A \sin 4A$$

$\boldsymbol{\sin(-A) = -\sin A}$

Formulae and Tables Book, p.13

12. $cos\,3x + cos\,9x$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos 3x + \cos 9x = 2\cos\frac{3x+9x}{2}\cos\frac{3x-9x}{2}$$
$$= 2\cos 6x \cos(-3x)$$
$$= 2\cos 6x \cos 3x$$

$\boldsymbol{\cos(-A) = \cos A}$

Formulae and Tables Book, p.13

Find the exact value of each of the following, without using a calculator.

13. $cos\,75° - cos\,15°$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos 75° - \cos 15° = -2\sin\frac{75°+15°}{2}\sin\frac{75°-15°}{2}$$
$$= -2\sin 45° \sin 30°$$
$$= -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$
$$= -\frac{1}{\sqrt{2}}$$
$$= -\frac{\sqrt{2}}{2}$$

14. $2\,sin\,60°\,sin\,120°$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$
$$2\sin 60° \cos 120°$$
$$= \cos(60° - 120°) - \cos(60° + 120°)$$
$$= \cos(-60°) - \cos 180°$$
$$= \cos(-60°) - \cos 180°$$
$$= \cos 60° - (-1)$$
$$= \frac{1}{2} + 1$$
$$= \frac{3}{2}$$

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15. $sin\,225° - sin\,15°$ **SKIP THIS ONE. Too long.**

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\begin{aligned}\sin 25° - \sin 15° &= 2\cos\frac{225°+15°}{2}\sin\frac{225°-15°}{2}\\ &= 2\cos 120°\sin 105°\\ &= 2(-\cos 60°)\sin 105°\\ &= 2\left(-\frac{1}{2}\right)\sin 105°\\ &= -\sin 105°\end{aligned}$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(60° + 45°) = \sin 60°\cos 45° + \cos 60°\sin 45°$$

$$\begin{aligned}\sin 105° &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

$$-\sin 105° = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

16. $2\,cos\,120°\,cos\,60°$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$\begin{aligned}2\cos 120°\cos 60° &= \cos(120° + 60°) + \cos(120° - 60°)\\ &= \cos 180° + \cos 60°\\ &= -1 + \frac{1}{2}\\ &= -\frac{1}{2}\end{aligned}$$

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17. $2sin\,45°\,sin\,15°$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$2\sin 45°\cos 15° = \cos(45°-15°) - \cos(45°+15°)$$
$$= \cos 30° - \cos 60°$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$
$$= \frac{\sqrt{3}-1}{2}$$

18. $cos\,37.5°\,sin\,37.5°$ **SKIP THIS ONE. Too long.**

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$\cos 37.5°\sin 37.5° = \frac{1}{2}[\sin(37.5°+37.5°) - \sin(37.5°-37.5°)]$$
$$= \frac{1}{2}[\sin 75° - \sin 0°]$$
$$= \frac{1}{2}[\sin 75° - 0]$$
$$= \frac{1}{2}\sin 75°$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(30°+45°) = \sin 30°\cos 45° + \cos 30°\sin 45°$$
$$\sin 75° = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$
$$\frac{1}{2}\sin 75° = \frac{1}{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$
$$= \frac{\sqrt{6}+\sqrt{2}}{8}$$

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Exercise 3

1. Prove that $tan\,\theta\,cos\,\theta = sin\,\theta$.

$$\tan\theta\cos\theta = \sin\theta$$
$$\left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta = \sin\theta$$
$$\sin\theta = \sin\theta$$
✓ True

2. Prove that $(cos\,A + sin\,A)^2 = sin\,2A + 1$.

$$(\cos A + \sin A)^2 = \sin 2A + 1$$
$$\cos^2 A + 2\cos A \sin A + \sin^2 A = \sin 2A + 1$$
$$2\cos A \sin A + 1 = \sin 2A + 1$$
$$2\cos A \sin A = \sin 2A$$
$$\sin 2A = \sin 2A$$ ✓ True

3. Prove that $sin\,2x = 2\,sin\,x\,cos\,x$.

$$\sin(A+B) = \sin A\cos B + \cos A \sin B$$
$$\sin(x+x) = \sin x\cos x + \cos x \sin x$$
$$\sin 2x = 2\sin x\cos x$$ ✓ True

4. Prove that $(sin\,\theta + cos\,\theta)^2 + (sin\,\theta - cos\,\theta)^2 = 2$.

$$(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$$
$$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 2$$
$$2\sin^2\theta + 2\cos^2\theta = 2$$
$$2(\sin^2\theta + \cos^2\theta) = 2$$
$$2(1) = 2$$ ✓ True

5. Prove that $tan\,\theta = \frac{sin\,2\theta}{cos\,2\theta+1}$.

$$\tan\theta = \frac{\sin 2\theta}{\cos 2\theta + 1}$$
$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta + 1}$$
$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta - (1 - \cos^2\theta) + 1}$$
$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta - 1 + \cos^2\theta + 1}$$
$$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$$
$$= \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta$$ ✓ True

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6. Prove that $sin(A+B)+sin(A-B)=2\,sin\,A\,cos\,B$.

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$
$$\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B=2\sin A\cos B$$
$$2\sin A\cos B=2\sin A\cos B \quad \checkmark \text{ True}$$

7. Prove that:

(a) $cos\,2x=cos^2x-sin^2x$.

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(x+x)=\cos x\cos x-\sin x\sin x$$
$$\cos 2x=\cos^2x-\sin^2x \quad \checkmark \text{ True}$$

(b) Deduce that $cos\,2x=2\,cos^2x-1$.

From (a): $\cos 2x=\cos^2x-\sin^2x$
$$=\cos^2x-(1-\cos^2x)$$
$$=\cos^2x-1+\cos^2x$$
$$=2\cos^2x-1 \quad \checkmark \text{ True}$$

(c) Hence, show that $cos^2x=\frac{1}{2}(1+cos\,2x)$.

From (b): $\cos 2x=2\cos^2x-1$
$$\cos 2x+1=2\cos^2x$$
$$\tfrac{1}{2}(\cos 2x+1)=\cos^2x \quad \checkmark \text{ True}$$

8. Prove that $tan\,2\theta=\frac{sin\,3\theta+sin\,\theta}{cos\,3\theta+cos\,\theta}$.

$$\tan 2\theta=\frac{\sin 3\theta+\sin\theta}{\cos 3\theta+\cos\theta}$$
$$=\frac{2\sin 2\theta\cos\theta}{2\cos 2\theta\cos\theta}$$
$$=\frac{\sin 2\theta}{\cos 2\theta}$$
$$=\tan 2\theta \quad \checkmark \text{ True}$$

$$\sin A+\sin B=2\sin\tfrac{A+B}{2}\cos\tfrac{A-B}{2}$$
$$\sin 3\theta+\sin\theta=2\sin\tfrac{3\theta+\theta}{2}\cos\tfrac{3\theta-\theta}{2}$$
$$=2\sin 2\theta\cos\theta$$

$$\cos A+\cos B=2\cos\tfrac{A+B}{2}\cos\tfrac{A-B}{2}$$
$$\cos 3\theta+\cos\theta=2\cos\tfrac{3\theta+\theta}{2}\cos\tfrac{3\theta-\theta}{2}$$
$$=2\cos 2\theta\cos\theta$$

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9. Prove that $cot\ 2x = \frac{sin\ x - sin\ 3x}{cos\ 3x - cos\ x}$.

$$\cot 2x = \frac{\sin x - \sin 3x}{\cos 3x - \cos x}$$
$$= \frac{-2\cos 2x \sin x}{-2\sin 2x \sin x}$$
$$= \frac{\cos 2x}{\sin 2x}$$
$$= \cot 2x \quad \checkmark \text{ True}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\sin x - \sin 3x = 2\cos\frac{x+3x}{2}\sin\frac{x-3x}{2}$$
$$= 2\cos 2x \sin(-x)$$
$$= -2\cos 2x \sin x$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos 3x - \cos x = -2\sin\frac{3x+x}{2}\sin\frac{3x-x}{2}$$
$$= -2\sin 2x \sin x$$

10. Prove that $4\ sin\ x\ cos^2\ x = sin\ 3x + sin\ x$.

$4\sin x \cos x\ (\cos x) = 2\sin 2x \cos x$

Divide both sides by $2\cos x$:

$$2\sin x \cos x = \sin 2x$$
$$\sin 2x = \sin 2x \quad \checkmark \text{ True}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin 3x + \sin x = 2\sin\frac{3x+x}{2}\cos\frac{3x-x}{2}$$
$$= 2\sin 2x \cos x$$

11. Prove that $sin\ 2A = \frac{2\ tan\ A}{1+tan^2\ A}$.

$$\sin 2A = \frac{2\left(\frac{\sin A}{\cos A}\right)}{1+\left(\frac{\sin A}{\cos A}\right)^2}$$
$$= \left(\frac{\frac{2\sin A}{\cos A}}{1+\frac{\sin^2 A}{\cos^2 A}}\right)\left(\frac{\cos^2 A}{\cos^2 A}\right)$$
$$= \frac{2\sin A \cos A}{\cos^2 A + \sin^2 A}$$
$$= \frac{2\sin A \cos A}{1}$$
$$= 2\sin A \cos A$$
$$= \sin 2A \quad \checkmark \text{ True}$$

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12. Prove that $cos\,3\theta = 4\,cos^3\,\theta - 3\,cos\,\theta$.

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$	$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$ $\cos 3\theta = (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$ $= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$ $= \cos^3\theta - 3\sin^2\theta\cos\theta$ $= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$ $= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$ $= 4\cos^3\theta - 3\cos\theta$ ✓ True

13. Show that $\frac{4}{sin^2\,2A} = \frac{1}{cos^2\,A} + \frac{1}{sin^2\,A}$.

$$\frac{4}{\sin^2 2A} = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}$$

$$\frac{4}{(\sin 2A)^2} = \frac{\sin^2 A + \cos^2 A}{\sin^2 A\cos^2 A}$$

$$\frac{4}{(2\sin A\cos A)^2} = \frac{1}{\sin^2 A\cos^2 A}$$

$$\frac{4}{4\sin^2 A\cos^2 A} = \frac{1}{\sin^2 A\cos^2 A}$$

$$\frac{1}{\sin^2 A\cos^2 A} = \frac{1}{\sin^2 A\cos^2 A}$$

✓ True

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14. (a) Show that $\frac{\sin 5x + \sin x}{\cos 5x + \cos x} = \tan 3x$.

$$\tan 3x = \frac{\sin 5x + \sin x}{\cos 5x + \cos x}$$
$$= \frac{2\sin 3x \cos 2x}{2\cos 3x \cos 2x}$$
$$= \frac{\sin 3x}{\cos 3x}$$
$$= \tan 3x \quad \checkmark \text{ True}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin 5x + \sin x = 2\sin\frac{5x+x}{2}\cos\frac{5x-x}{2}$$
$$= 2\sin 3x \cos 2x$$

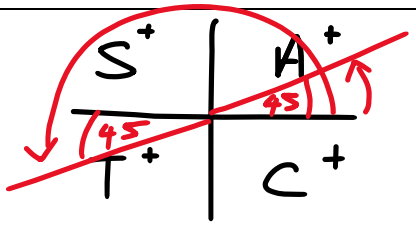
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos 5x + \cos x = 2\cos\frac{5x+x}{2}\cos\frac{5x-x}{2}$$
$$= 2\cos 3x \cos 2x$$

(b) Hence, solve $\frac{\sin 5x + \sin x}{\cos 5x + \cos x} = 1$, for **(i)** $0 \leq x \leq 360°$.

(ii) $x \in \mathbb{R}$. (Find the general solution for x.)

From (a): $\tan 3x = \frac{\sin 5x + \sin x}{\cos 5x + \cos x}$

$$\tan 3x = 1$$
$$\tan R = 1$$
$$R = 45°$$



(i) $0 \leq x \leq 360°$.

$0 \leq 3x \leq 1080°$. 3 full turns. 6 solutions.

$3x = 45°, 225°, 405°, 585°, 765°, 945°$

+360 +360

$x = 15°, 75°, 135°, 195°, 255°, 315°$

(ii) $x \in \mathbb{R}$. General solution.

$3x = 45° + 360°n, 225° + 360°n$

$x = 15° + 120°n, 75° + 120°n$

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