THE INSTITUTE OF EDUCATION

Maths

Stephanie Mulligan

Leaving Cert

Higher Level

Trigonometry
– Trig Identities and Proofs

STUDENT NAME:

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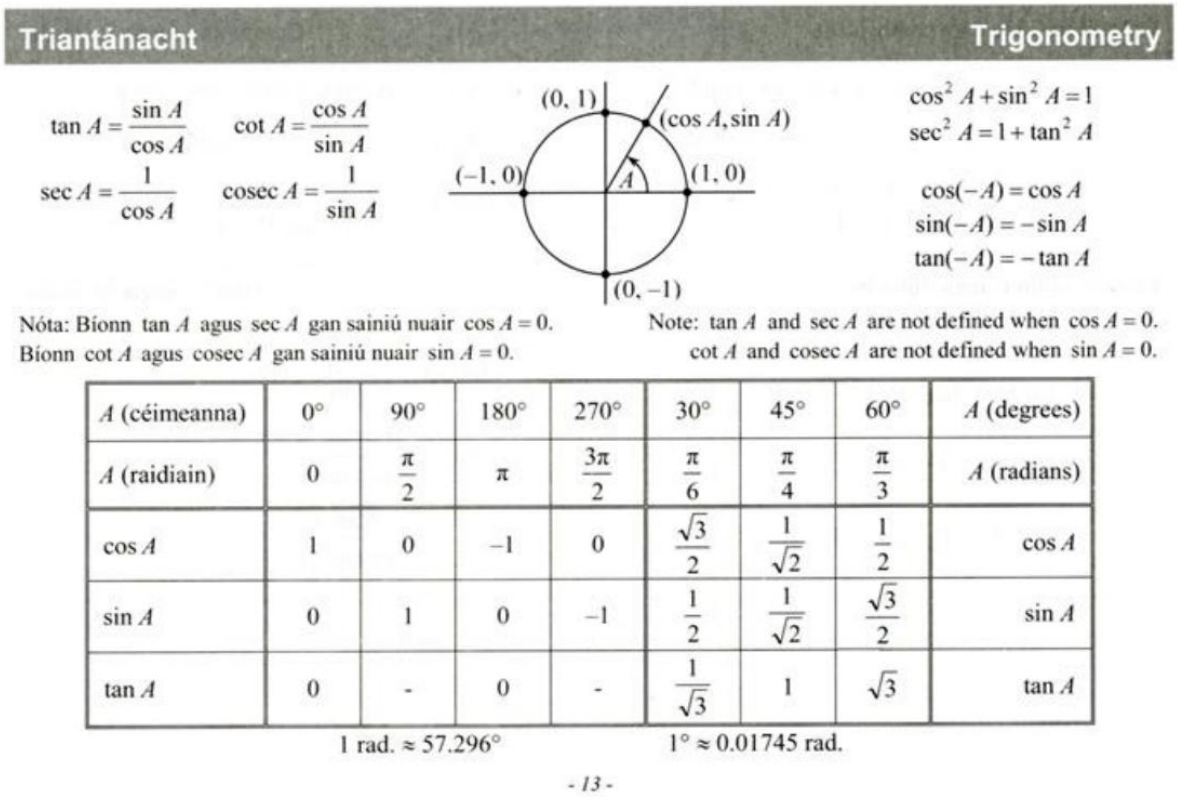
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Special Angles in Log Tables p.13 (Log Tables)



Triantánacht **Trigonometry**

$\tan A = \frac{\sin A}{\cos A}$ $\cot A = \frac{\cos A}{\sin A}$

$\sec A = \frac{1}{\cos A}$ $\operatorname{cosec} A = \frac{1}{\sin A}$

$\cos^2 A + \sin^2 A = 1$

$\sec^2 A = 1 + \tan^2 A$

$\cos(-A) = \cos A$

$\sin(-A) = -\sin A$

$\tan(-A) = -\tan A$

Nóta: Bíonn $\tan A$ agus $\sec A$ gan sainiú nuair $\cos A = 0$.
Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

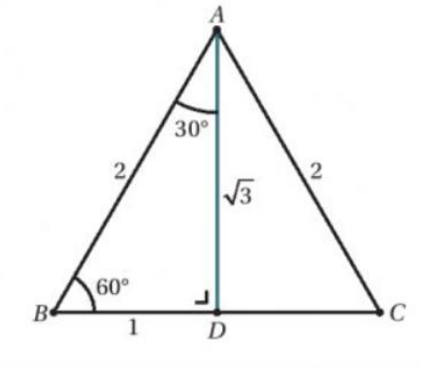
A (céimeanna)	0°	90°	180°	270°	30°	45°	60°	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	−1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	−1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. ≈ 57.296° 1° ≈ 0.01745 rad.

- 13 -

60° and 30°

$\sin 60°$, $\cos 60°$, $\tan 60°$ and $\sin 30°$, $\cos 30°$, $\tan 30°$ can be found using an equilateral triangle with side lengths of 2 units.



- In an equilateral triangle, the perpendicular bisector of a side also bisects the angle across from that side (as seen in Constructions).
- In the diagram, $AD \perp BC$ and $|BD| = 1$.
  So, $|\angle BAD| = 30°$.
- By Pythagoras, $|AD| = \sqrt{3}$.

This gives:

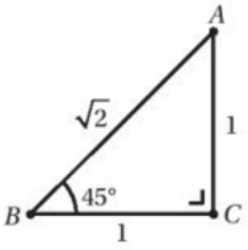
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$	$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$
$\mathbf{\sin 60° = \frac{\sqrt{3}}{2}}$	$\mathbf{\cos 60° = \frac{1}{2}}$	$\mathbf{\tan 60° = \sqrt{3}}$
$\mathbf{\sin 30° = \frac{1}{2}}$	$\mathbf{\cos 30° = \frac{\sqrt{3}}{2}}$	$\mathbf{\tan 30° = \frac{1}{\sqrt{3}}}$

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45°

$\sin 45°, \cos 45°, \tan 45°$ can be found using an isoceles right-angled triangle with equal side lengths of 1 unit.

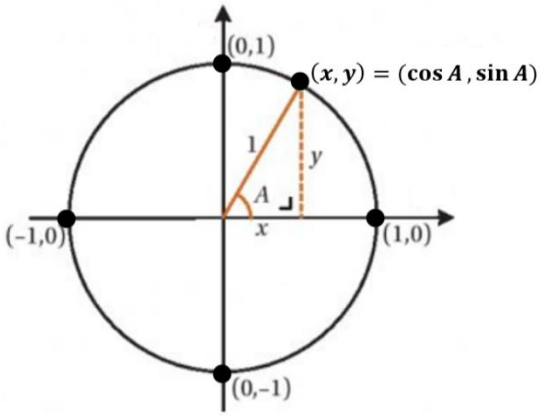


- By Pythagoras, $|AB| = \sqrt{2}$.

This gives:

$\mathbf{\sin 45° = \frac{1}{\sqrt{2}}}$	$\mathbf{\cos 45° = \frac{1}{\sqrt{2}}}$	$\mathbf{\tan 45° = 1}$

0°, 90°, 180°, 270°



$\sin A$ and $\cos A$ are defined for all angles by The Unit Circle.

- The Unit Circle has centre $(0, 0)$ and radius 1 unit.
- For any point on the circumference $(x, y) = (cos\, A, sin\, A)$, where A is the angle of rotation from the positive sense of the x-axis (where the number 3 is on a clock).
- Rotate anticlockwise for positive angles and clockwise for negative angles.

The right-angled triangle shown in the diagram gives:

$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$	$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$
$\sin A = \frac{y}{1}$	$\cos A = \frac{x}{1}$	$\tan A = \frac{y}{x}$
$\mathbf{\sin A = y}$	$\mathbf{\cos A = x}$	$\mathbf{\tan A = \frac{\sin A}{\cos A}}$

That is, for any point on the circumference of The Unit Circle, the x-coordinate is equal to $\cos A$ and the y-coordinate is equal to $\sin A$ and $\tan A = \frac{\cos A}{\sin A}$, as per page 13 of the log tables.

$\boldsymbol{A}$	$\mathbf{0°}$	$\mathbf{90°}$	$\mathbf{180°}$	$\mathbf{270°}$
$\mathbf{\cos A}$	1	0	-1	0
$\mathbf{\sin A}$	0	1	0	-1
$\mathbf{\tan A = \frac{\sin A}{\cos A}}$	$\frac{1}{0} \rightarrow$ Undefined	$\frac{0}{1} = 0$	$\frac{-1}{0} \rightarrow$ Undefined	$\frac{0}{-1} = 0$

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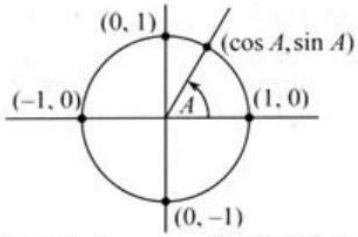
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Using Trig Identities (Log Tables pages 13, 14 and 15)

Triantánacht **Trigonometry**

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$



$$\cos^2 A + \sin^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$

$$\cos(-A) = \cos A$$
$$\sin(-A) = -\sin A$$
$$\tan(-A) = -\tan A$$

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Bíonn $\cot A$ agus $\operatorname{cosec} A$ gan sainiú nuair $\sin A = 0$.

Note: $\tan A$ and $\sec A$ are not defined when $\cos A = 0$.
$\cot A$ and $\operatorname{cosec} A$ are not defined when $\sin A = 0$.

A (céimeanna)	0°	90°	180°	270°	30°	45°	60°	A (degrees)
A (raidiain)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	A (radians)
$\cos A$	1	0	−1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\cos A$
$\sin A$	0	1	0	−1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\sin A$
$\tan A$	0	-	0	-	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\tan A$

1 rad. ≈ 57.296° 1° ≈ 0.01745 rad.

Foirmlí uillinneacha comhshuite **Compound angle formulae**

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Foirmlí uillinneacha dúbailte **Double angle formulae**

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\sin 2A = 2\sin A\cos A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A) \qquad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A) \qquad \sin 2A = \frac{2\tan A}{1 + \tan^2 A}$$

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Iolraigh a thiontú ina suimeanna agus ina ndifríochtaí — **Products to sums and differences**

$$2\cos A\cos B = \cos(A+B)+\cos(A-B)$$

$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$

$$2\cos A\sin B = \sin(A+B)-\sin(A-B)$$

Suimeanna agus difríochtaí a thiontú ina n-iolraigh — **Sums and differences to products**

$$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A-\cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

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Using the Compound Angle Formulae

There are six compound angle formulae on the top of page 14 of the Formulae and Tables Book.

$\cos(A+B) = \cos A \cos B - \sin A \sin B$	$\cos(A-B) = \cos A \cos B + \sin A \sin B$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$	$\sin(A-B) = \sin A \cos B - \cos A \sin B$
$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$	$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

We can use these to work values for $\sin\theta$, $\cos\theta$ and $\tan\theta$, **without a calculator**, where θ can be written as a combination of the special angles on page 13 of the Formulae and Tables Book.
For example, $\theta = 75°$ can be written as $\theta = (30° + 45°)$.

Example 1 Express each of the following in surd form, without the use of a calculator.
(a) $\sin 75°$ **(b)** $\tan 15°$

Solution

(a) $sin\,75° = sin(30° + 45°)$

$$sin(A+B) = sin\,A\,cos\,B + cos\,A\,sin\,B$$
$$sin(30+45) = sin\,30\,cos\,45 + cos\,30\,sin\,45$$
$$sin\,75° = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

(b) $tan\,15° = tan(45° - 30°)$

$$tan(A-B) = \frac{tan\,A - tan\,B}{1 + tan\,A\,tan\,B}$$
$$tan(45-30) = \frac{tan\,45 - tan\,30}{1 + tan\,45\,tan\,30}$$
$$tan\,15° = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\left(\frac{1}{\sqrt{3}}\right)}$$
$$= \left(\frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\left(\frac{1}{\sqrt{3}}\right)}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$
$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$
$$= 2 - \sqrt{3}$$

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Example 2 Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{2}{\sqrt{13}}$, find the exact value of $\sin(A + B)$, without the use of a calculator. $0° < A < 90°$ and $0° < B < 90°$.

Solution

$$sin(A + B) = sin\, A\, cos\, B + cos\, A\, sin\, B$$

$$sin(A + B) = \left(\frac{3}{5}\right)\left(\frac{2}{\sqrt{13}}\right) + \left(\frac{4}{5}\right)\left(\frac{3}{\sqrt{13}}\right)$$

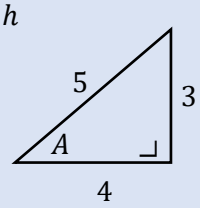
$$= \frac{18}{5\sqrt{13}}$$

We have $\sin A$, we need $\cos A$.
We have $\cos B$, we need $\sin B$.

By Pythagoras:

$$\sin A = \frac{3}{5} = \frac{o}{h}$$

$$h^2 = o^2 + a^2$$
$$5^2 = 3^2 + a^2$$
$$\sqrt{25 - 9} = a$$
$$4 = a$$

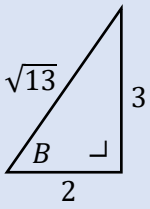


$$\cos A = \frac{a}{h} = \frac{4}{5}$$

By Pythagoras:

$$\cos B = \frac{2}{\sqrt{13}} = \frac{a}{h}$$

$$h^2 = o^2 + a^2$$
$$\left(\sqrt{13}\right)^2 = o^2 + 2^2$$
$$\sqrt{13 - 4} = o$$
$$3 = o$$



$$\sin B = \frac{o}{h} = \frac{3}{\sqrt{13}}$$

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Using the Double Angle Formulae

There are seven double angle formulae on the bottom of page 14 of the Formulae and Tables Book.

$\cos 2A = \cos^2 A - \sin^2 A$

$\sin 2A = 2\sin A \cos A$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$

We can use these to work values for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$, **without a calculator**, when given values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ (or vice versa).

Example 1 $\cos A = \frac{4}{5}$, where $0° < A < 90°$. Evaluate, without the use of a calculator:
(a) $\cos 2A$ **(b)** $\sin 2A$

Solution

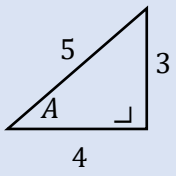
(a) $cos\,2A = cos^2 A - sin^2 A$

We know $\cos A = \frac{4}{5}$, we need $\sin A$.

By Pythagoras:

$$\cos A = \frac{4}{5} = \frac{a}{h}$$

$$h^2 = o^2 + a^2$$
$$5^2 = o^2 + 4^2$$
$$\sqrt{25 - 16} = o$$
$$3 = o$$



$$\sin A = \frac{o}{h} = \frac{3}{5}$$

$$cos\,2A = cos^2 A - sin^2 A$$
$$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$$
$$= \frac{7}{25}$$

(b) $sin\,2A = 2\,sin\,A\,cos\,A$
$$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$$
$$= \frac{24}{25}$$

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Example 2 Given $\sin 2\theta = \frac{3}{5}$, find two values of $\tan\theta$, where $0° < \theta < 90°$.

Solution

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\frac{3}{5} = \frac{2\tan\theta}{1+\tan^2\theta}$$

Let $x = \tan\theta$:

$$\frac{3}{5} = \frac{2x}{1+x^2}$$

$$3(1+x^2) = 5(2x)$$

$$3 + 3x^2 = 10x$$

$$3x^2 - 10x + 3 = 0$$

$$(3x-1)(x-3) = 0$$

$$x = \frac{1}{3}, \quad x = 3$$

Swap back:

$$\tan\theta = \frac{1}{3}, \tan\theta = 3$$

Example 3 A and B are acute angles such that $\tan B = \frac{1}{4}$ and $\tan(A-B) = 2$.
Without evaluating A or B, work out the value of: **(a)** $\tan A$ **(b)** $\cos 2A$.

Solution

(a)

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$2 = \frac{\tan A - \frac{1}{4}}{1 + \tan A\left(\frac{1}{4}\right)}$$

Let $\tan A = x$:

$$2 = \frac{x - \frac{1}{4}}{1 + x\left(\frac{1}{4}\right)}\left(\frac{4}{4}\right)$$

$$2 = \frac{4x-1}{4+x}$$

$$2(4+x) = 4x - 1$$

$$8 + 2x = 4x - 1$$

$$9 = 2x$$

$$\frac{9}{2} = x$$

Swap back: $\tan A = \frac{9}{2}$

(b)

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$= \frac{1-\left(\frac{9}{2}\right)^2}{1+\left(\frac{9}{2}\right)^2}$$

$$= -\frac{77}{85}$$

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Exercise 1

1. Express each of the following in surd form, without using a calculator.

(a) $sin\,75°$ **(b)** $cos\,105°$ **(c)** $tan\,75°$

(d) $cos\,15°$ **(e)** $tan\,105°$ **(f)** $sin\,165°$

2. A and B are acute angles. $cos\,A = \frac{5}{13}$ and $sin\,B = \frac{3}{5}$. Work out the value of each of the following, without using a calculator.

(a) $sin\,A$ **(b)** $tan\,A$ **(c)** $cos\,B$

(d) $tan\,B$ **(e)** $sin(A - B)$ **(f)** $cos(A - B)$

(g) $tan(A + B)$ **(h)** $sin\,2A$ **(i)** $tan\,2A$

3. A and B are acute angles. $sin\,A = \frac{15}{17}$ and $cos\,B = \frac{3}{\sqrt{13}}$. Work out the value of each of the following, without using a calculator. Where appropriate, give your answer in surd form.

(a) $sin(A + B)$ **(b)** $cos\,2A$ **(c)** $tan(A - B)$

4. $tan\,X = \frac{4}{3}$, where $0° < X < 90°$. Evaluate, without evaluating X.

(a) $sin\,2X$ **(b)** $cos\,2X$ **(c)** $tan\,2X$

5. Given that $tan\,\theta = \frac{1}{3}$, show that $tan\,2\theta = \frac{3}{4}$.

6. If $cos\,\theta = \frac{1}{\sqrt{3}}$, work out the value of $cos\,2\theta$, without evaluating θ.

7. $tan\,A = 4$ and $tan(A + B) = 5$, where $0° < A < 90°$ and $0° < B < 90°$.
Work out the value of $tan\,B$.

8. $tan\,A = 2$ and $tan(A + B) = 3$, where $0° < A < \frac{\pi}{2}$ and $0° < B < \frac{\pi}{2}$.
Work out the value of $tan\,B$.

9. A and B are acute angles such that $tan\,B = \frac{1}{3}$ and $tan(A - B) = 2$. Without evaluating A or B, work out the value of: **(a)** $tan\,A$ **(b)** $cos\,2A$.

10. If $sin\,2A = \frac{15}{17}$ and $0° < A < 90°$. Find two possible values of $tan\,A$, without evaluating A.

11. If $sin\,2X = \frac{24}{25}$ and $0° < X < 90°$. Find two possible values of $tan\,X$, without evaluating X.

12. If $cos\,2B = \frac{12}{13}$ and $0° < B < \frac{\pi}{2}$. Find the value of $cos\,B$, without evaluating B.

13. If $cos\,2\theta = \frac{7}{25}$ and $0° < \theta < \frac{\pi}{2}$. Find the value of $cos\,\theta$, without evaluating θ.

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Answers:

1. **(a)** $\frac{\sqrt{6}+\sqrt{2}}{4}$ **(b)** $\frac{\sqrt{2}-\sqrt{6}}{4}$ **(c)** $2+\sqrt{3}$

(d) $\frac{\sqrt{6}+\sqrt{2}}{4}$ **(e)** $-2-\sqrt{3}$ **(f)** $\frac{\sqrt{6}-\sqrt{2}}{4}$

2. **(a)** $\frac{12}{13}$ **(b)** $\frac{12}{5}$ **(c)** $\frac{4}{5}$

(d) $\frac{3}{4}$ **(e)** $\frac{33}{65}$ **(f)** $\frac{56}{65}$

(g) $-\frac{63}{16}$ **(h)** $\frac{120}{169}$ **(i)** $-\frac{120}{119}$

3. **(a)** $\frac{61}{17\sqrt{13}}$ or $\frac{61\sqrt{13}}{221}$ **(b)** $-\frac{161}{289}$ **(c)** $\frac{29}{54}$

4. **(a)** $\frac{24}{25}$ **(b)** $-\frac{7}{25}$ **(c)** $-\frac{24}{7}$

5. $\frac{3}{4}$ **6.** $-\frac{1}{3}$ **7.** $\frac{1}{21}$

8. $\frac{1}{7}$ **9.** **(a)** 7 **(b)** $-\frac{24}{25}$

10. $\frac{5}{3}$ and $\frac{3}{5}$ **11.** $\frac{4}{3}$ and $\frac{3}{4}$ **12.** $\frac{5}{\sqrt{26}}$

13. $\frac{4}{5}$

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Using the Product, Sum and Difference Formulae

On page 14 of the Formulae and Tables Book there are four formulae to convert products of trig ratios to sums and differences and four formulae to convert sums and differences to products.

Products to Sums and Differences

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

Sums and Differences to Products

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

Example 1 Express each of the following expressions as a sum or difference.

(a) $2\sin 3x\cos x$ **(b)** $\sin 3\theta\sin 7\theta$

Solution

(a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\sin 3x\cos x = \sin(3x+x) + \sin(3x-x)$

$= \sin 4x + \sin 2x$

(b) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$\sin 3\theta\sin 7\theta = \frac{1}{2}(\cos(3\theta-7\theta) - \cos(3\theta+7\theta))$

$= \frac{1}{2}(\cos(-4\theta) - \cos 10\theta)$

$= \frac{1}{2}(\cos(4\theta) - \cos 10\theta)$

$\cos(-A) = \cos A$

Formulae and Tables Book, p.13

Example 2 Express each of the following expressions as a product.

(a) $\cos 3x + \cos x$ **(b)** $\sin\left(\theta+\frac{\pi}{2}\right) + \sin\left(\theta-\frac{\pi}{2}\right)$

Solution

(a) $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos 3x + \cos x = 2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}$

$= 2\cos 2x\cos x$

(b) $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\sin\left(\theta+\frac{\pi}{2}\right) - \sin\left(\theta-\frac{\pi}{2}\right) = 2\cos\frac{\left(\theta+\frac{\pi}{2}\right)+\left(\theta-\frac{\pi}{2}\right)}{2}\sin\frac{\left(\theta+\frac{\pi}{2}\right)-\left(\theta-\frac{\pi}{2}\right)}{2}$

$= 2\cos\frac{2\theta}{2}\sin\frac{\pi}{2}$

$= 2\cos\theta\,(1)$

$= 2\cos\theta$

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Exercise 2

Write each of the following as a sum or difference of two trigonometric functions.

1. $2\sin 5x\cos 3x$ **2.** $2\cos 7y\cos y$ **3.** $2\cos 6\theta\cos 2\theta$

4. $-2\sin 5A\sin A$ **5.** $\cos x\sin 6x$ **6.** $\cos 5B\sin B$

Write each of the following as a product of two trigonometric functions.

7. $\sin 6x+\sin 2x$ **8.** $\cos 10\theta-\cos 4\theta$ **9.** $\sin 7y-\sin 3y$

10. $\cos 2x+\cos 6x$ **11.** $\sin 4A-\sin 12A$ **12.** $\cos 3x+\cos 9x$

Find the exact value of each of the following, without using a calculator.

13. $\cos 75°-\cos 15°$ **14.** $2\sin 60°\sin 120°$ **15.** $\sin 225°-\sin 15°$

16. $2\cos 120°\cos 60°$ **17.** $2\sin 45°\sin 15°$ **18.** $\cos 37.5°\sin 37.5°$

Replace

Answers:

1. $\sin 8x+\sin 2x$ **2.** $\cos 8y+\cos 6y$ **3.** $\cos 8\theta+\cos 4\theta$

4. $\cos 6A-\cos 4A$ **5.** $\frac{1}{2}(\sin 7x+\sin 5x)$ **6.** $\frac{1}{2}(\sin 6B-\sin 4B)$

7. $2\sin 4x\cos 2x$ **8.** $-2\sin 7\theta\sin 3\theta$ **9.** $2\cos 5y\sin 2y$

10. $2\cos 4x\cos 2x$ **11.** $-2\cos 8A\sin 4A$ **12.** $2\cos 6x\cos 3x$

13. $-\frac{\sqrt{2}}{2}$ **14.** $\frac{3}{2}$ **15.** $-\frac{\sqrt{6}+\sqrt{2}}{4}$

16. $-\frac{1}{2}$ **17.** $\frac{\sqrt{3}-1}{2}$ **18.** $\frac{\sqrt{6}+\sqrt{2}}{8}$

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Forming Proofs with Trig Identities

Solving problems such as: Show that $\frac{\cos 7A + \cos A}{\sin 7A - \sin A} = \cot 3A$ (Solution below)

Usually involve:

- Making one or two (maybe three) substitutions of terms or expressions in the given equation with their equivalent forms, found in the trig identities on pages 13, 14 and 15 of the log table.
- Manipulating the equation with algebraic skills.

There are multiple approaches to these proofs and all are equally excellent. Do whatever feels right for you. Some of these approaches are:

- Work on the left-hand side and right-hand side separately to bring each to the same conclusion.
- Work on one side of the equation only, to arrive at the statement on the other side.
- Work on the equation in its entirety, manipulating it to arrive at a true statement.
  Note that the true statement (which you can tick and say "true") can be another identity from pages 13, 14 and 15 of the log tables.

Of the above approaches, I use the third one most often but I do interchange between them, depending on the scenario. I do whatever feels right and seems like the least effort.

The best advice I can give with these is to use your instinct and do whatever you see can be done. Don't worry about whether what you're doing is "the right way". There are many right ways.

In the leaving cert exam, the marking scheme for these tends to be very kind and lots of marks are rewarded for "any relevant work" or simply making one correct substitution from pages 13, 14 and 15 of the log tables. An example of this is shown below.

Example 1

2016 HL LC Maths Paper 2 - Question 3 (a)

Show that $\frac{\cos 7A + \cos A}{\sin 7A - \sin A} = \cot 3A$

Solution and Marking Scheme

$\frac{2\cos\frac{7A+A}{2}\cos\frac{7A-A}{2}}{2\cos\frac{7A+A}{2}\sin\frac{7A-A}{2}}$ $\frac{2\cos 4A\cos 3A}{2\cos 4A\sin 3A}$ $= \frac{\cos 3A}{\sin 3A}$ $= \cot 3A$	Scale 15C (0, 5, 10, 15) *Low Partial Credit* • sum to product formula with some substitution *High Partial Credit* • sum to product formula fully substituted

The identities used on the first line are at the bottom of page 15 of the log tables:

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

The identity used in the last line is on the top left of page 13 of the log tables:

$$\cot A = \frac{\cos A}{\sin A}$$

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Exercise 3

1. Prove that $tan\,\theta\,cos\,\theta = sin\,\theta$.

2. Prove that $(cos\,A + sin\,A)^2 = sin\,2A + 1$.

3. Prove that $sin\,2x = 2\,sin\,x\,cos\,x$.

4. Prove that $(sin\,\theta + cos\,\theta)^2 + (sin\,\theta - cos\,\theta)^2 = 2$.

5. Prove that $tan\,\theta = \frac{sin\,2\theta}{cos\,2\theta+1}$.

6. Prove that $sin(A + B) + sin(A - B) = 2\,sin\,A\,cos\,B$.

7. Prove that:

(a) $cos\,2x = cos^2\,x - sin^2\,x$.

(b) Deduce that $cos\,2x = 2\,cos^2\,x - 1$.

(c) Hence, show that $cos^2\,x = \frac{1}{2}(1 + cos\,2x)$.

8. Prove that $tan\,2\theta = \frac{sin\,3\theta+sin\,\theta}{cos\,3\theta+cos\,\theta}$.

9. Prove that $cot\,2x = \frac{sin\,x-sin\,3x}{cos\,3x-cos\,x}$.

10. Prove that $4\,sin\,x\,cos^2\,x = sin\,3x + sin\,x$.

11. Prove that $sin\,2A = \frac{2\,tan\,A}{1+tan^2\,A}$.

12. Prove that $cos\,3\theta = 4\,cos^3\,\theta - 3\,cos\,\theta$.

13. Show that $\frac{4}{sin^2\,2A} = \frac{1}{cos^2\,A} + \frac{1}{sin^2\,A}$.

14. **(a)** Show that $\frac{sin\,5x + sin\,x}{cos\,5x+cos\,x} = tan\,3x$.

(b) Hence, solve $\frac{sin\,5x + sin\,x}{cos\,5x+cos\,x} = 1$, for **(i)** $0 \leq x \leq 360°$.

(ii) $x \in \mathbb{R}$. (Find the general solution for x.)

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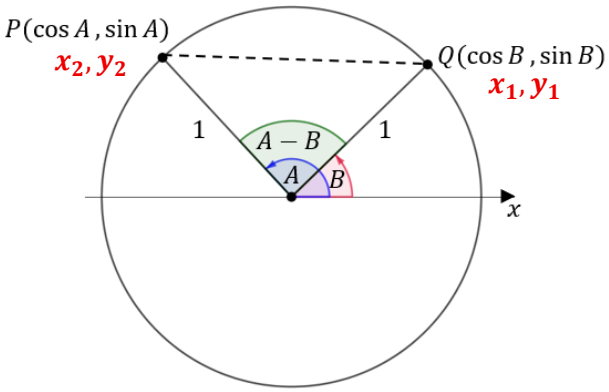
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Formal Proofs of Trig Identities – Learn by Heart

The formal proofs of the six compound angle formulae on page 14 of The Formulae and Tables book are examinable and need to be committed to memory. The proof of $\cos 2A$ should also be learned by heart.

1. Prove $\cos(A-B) = \cos A \cos B + \sin A \sin B$

Let $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$ be two points of the unit circle.
Radii $|OP| = |OQ| = 1$.



The diagram is required for full marks.

3 Steps
1. Distance formula on $[PQ]$ to find $|PQ|^2$.
2. Cosine Rule to find $|PQ|^2$.
3. Equate $|PQ|^2$ from step 1 with $|PQ|^2$ from step 2 and simplify.

1. Distance Formula

$$|PQ| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$
$$|PQ|^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$$
$$= \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A \cos B - 2\sin A \sin B$$
$$= 1 + 1 - 2\cos A \cos B - 2\sin A \sin B$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B)$$

$\boldsymbol{\cos^2 A + \sin^2 A = 1}$
Formulae and Tables Book, p.13

2. Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$|PQ|^2 = 1^2 + 1^2 - 2(1)(1)\cos(A-B)$$
$$= 2 - 2\cos(A-B)$$

3. Equate

$$|PQ|^2 = |PQ|^2$$
$$2 - 2\cos(A-B) = 2 - 2(\cos A \cos B + \sin A \sin B)$$
$$\boldsymbol{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$ ✓😊

2. Prove $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Use identity $\cos(A-B) = \cos A \cos B + \sin A \sin B$ and replace B with $-B$.

$$\cos[A-(-B)] = \cos A \cos(-B) + \sin A \sin(-B)$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$ ✓😊

$\boldsymbol{\cos(-A) = \cos A}$
$\boldsymbol{\sin(-A) = -\sin A}$
Formulae and Tables Book, p.13

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3. Prove $\sin(A+B) = \sin A \cos B + \cos A \sin B$

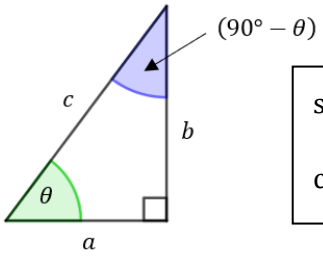
Use identity $\cos(A-B) = \cos A \cos B + \sin A \sin B$ and replace A with $(90° - A)$.

$\cos[(90° - A) - B] = \cos(90° - A)\cos B + \sin(90° - A)\sin B$
$\cos[90° - (A+B)] = \sin A \cos B + \cos A \sin B$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$ ✓😊

Note on Proof (You do not need to write this in an exam)

$\cos(90° - A) = \sin A$

$\sin(90° - A) = \cos A$



$\sin\theta = \frac{b}{c}$ $\quad$ $\sin(90° - \theta) = \frac{a}{c}$

$\cos\theta = \frac{a}{c}$ $\quad$ $\cos(90° - \theta) = \frac{b}{c}$

4. Prove $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Use identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and replace B with $-B$.

$\sin[A + (-B)] = \sin A \cos(-B) + \cos A \sin(-B)$
$\sin(A-B) = \sin A \cos B - \cos A \sin B$ ✓😊

$\sin(-A) = -\sin A$

Formulae and Tables Book, p.13

5. Prove $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Divide top and bottom by $\cos A \cos B$

$$= \frac{\frac{\sin A \cos B}{\sin A \cos B} + \frac{\cos A \sin B}{\sin A \cos B}}{\frac{\cos A \cos B}{\sin A \cos B} - \frac{\sin A \sin B}{\sin A \cos B}}$$

(no need to write this line in an exam)

$$= \frac{\frac{\sin A}{\cos B} + \frac{\sin B}{\cos B}}{1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$ ✓😊

$\tan A = \frac{\sin A}{\cos A}$

Formulae and Tables Book, p.13

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6. Prove $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

Use identity $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$ and replace B with $-B$.

$$\tan[A+(-B)]=\frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$ ✓😊

$\tan(-A)=-\tan A$

Formulae and Tables Book, p.13

7. Prove $\cos 2A=\cos^2 A-\sin^2 A$

Use $\cos(A+B)=\cos A\cos B-\sin A\sin B$ and replace B with A.

$$\cos(A+A)=\cos A\cos A-\sin A\sin A$$

$$\cos 2A=\cos^2 A-\sin^2 A$$ ✓😊

8. Prove $\sin 2A=2\sin A\cos A$

Use $\sin(A+B)=\sin A\cos B+\cos A\sin B$ and replace B with A.

$$\sin(A+A)=\sin A\cos A+\cos A\sin A$$

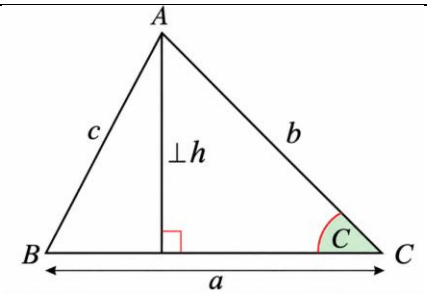
$$\sin 2A=2\sin A\cos A$$ ✓😊

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Proof of Area of Triangle Formula, Sine Rule and Cosine Rule

Prove: Area of $\triangle = \frac{1}{2}\boldsymbol{ab\sin C}$



Area of $\triangle ABC = \frac{1}{2}$ base $\times$ perpendicular height

$= \frac{1}{2}a(\perp h)$

$= \frac{1}{2}ab\sin C$

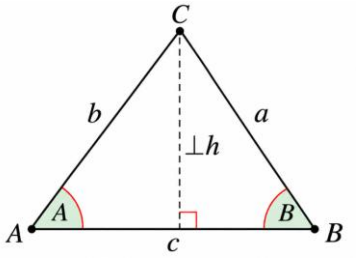
$\sin B = \frac{\text{opposite}}{\text{hypotenuse}}$

$\sin C = \frac{\perp h}{b}$

$b\sin C = \perp h$

Prove Sine Rule: $\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$

Acute Triangle:



$\sin A = \frac{\perp h}{b}$

$b\sin A = \perp h$

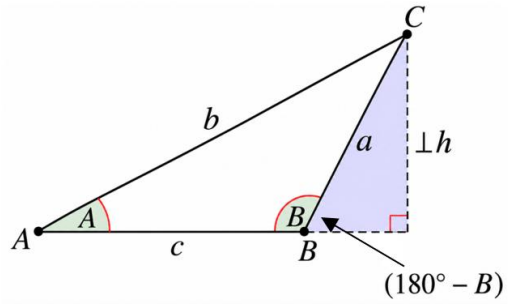
$\sin B = \frac{\perp h}{a}$

$a\sin B = \perp h$

$\perp h = \perp h$

$a\sin B = b\sin A$

Obtuse Triangle:



$\sin A = \frac{\perp h}{b}$

$b\sin A = \perp h$

$\sin(180° - B) = \frac{\perp h}{a}$

$\sin B = \frac{\perp h}{a}$

$a\sin B = \perp h$

$\perp h = \perp h$

$a\sin B = b\sin A$

In both cases, dividing by $\sin A \sin B$ results in:

$$\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B}}$$

Similarly, it can be shown that:

$$\boldsymbol{\frac{b}{\sin B} = \frac{c}{\sin C}}$$

Therefore:

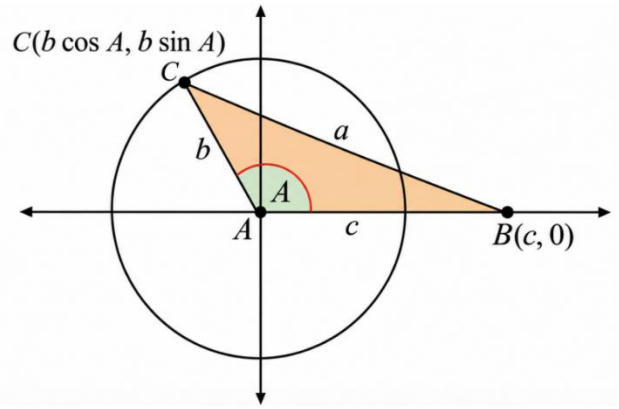
$$\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

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Prove Cosine Rule: $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$

Step 1: Diagram



See unit circle note below for point C label explanation.

Step 2: Distance Formula on $|BC|$

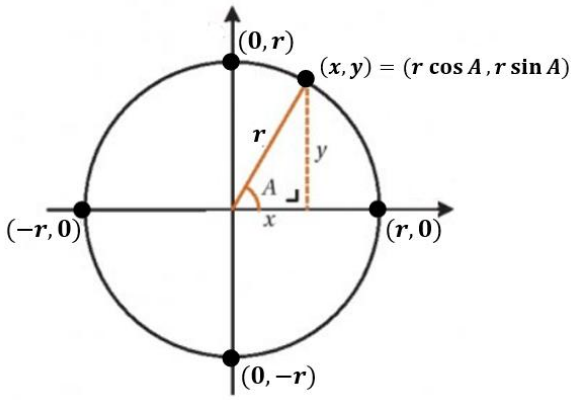
$$|BC| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$|BC|^2 = (c - b\cos A)^2 + (0 - b\sin A)^2$$
$$a^2 = c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin^2 A$$
$$a^2 = c^2 - 2bc\cos A + b^2(\cos^2 A + \sin^2 A)$$
$$a^2 = c^2 - 2bc\cos A + b^2(1)$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

$\boldsymbol{\cos^2 A + \sin^2 A = 1}$

Formulae and Tables Book, p.13

Note:

Changing the radius of The Unit Circle from 1 unit to r units gives $\boldsymbol{(x, y) = (r\cos A, r\sin A)}$, as follows:



$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$	$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$
$\sin A = \frac{y}{r}$	$\cos A = \frac{x}{r}$	$\tan A = \frac{y}{x}$
$\boldsymbol{r\sin A = y}$	$\boldsymbol{r\cos A = x}$	$\boldsymbol{\tan A = \frac{r\sin A}{r\cos A} = \frac{\sin A}{\cos A}}$

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