Maths

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Higher Level

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Algebra - Powers & Logs

Unit 8

Algebra

Powers and Logs

8.1 Powers and the Laws of Indices

1. **Definition of Powers and Indices**
In the statement

$$2^3 = 8$$

* 2^3 is called a **power**, i.e. '2 to the power of 3',
* 2 is the **base** of the power,
* 3 is the **index**, or **exponent**, of the power,
* 8 is the **value** of the power.

2. **Laws of Indices**
The rules for dealing with powers are called the Laws of Indices. They are given on page 21 of the *Formulae and Tables*.

Laws of Indices

1.	$a^p.a^q = a^{p+q}$,	e.g. $3^5.3^2 = 3^{5+2} = 3^7$
2.	$\dfrac{a^p}{a^q} = a^{p-q}$	e.g. $\dfrac{4^8}{4^3} = 4^{8-3} = 4^5$
3.	$\left(a^p\right)^q = a^{pq}$	e.g. $\left(x^2\right)^{n+1} = x^{2(n+1)} = x^{2n+2}$
4.	$a^0 = 1$	e.g. $3^0 = 1$
5.	$a^{-p} = \dfrac{1}{a^p}$	e.g. $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$
6.	$a^{\frac{1}{q}} = \sqrt[q]{a}$	e.g. $a^{\frac{1}{3}} = \sqrt[3]{a}$
7.	$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$	e.g. $27^{\frac{4}{3}} = \sqrt[3]{27^4} = \left(\sqrt[3]{27}\right)^4$
8.	$(ab)^p = a^p b^p$	e.g. $(3x)^4 = 3^4 x^4 = 81x^4$
9.	$\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$	e.g. $\left(\dfrac{2x}{y}\right)^4 = \dfrac{(2x)^4}{y^4} = \dfrac{16x^4}{y^4}$

Example 1

Express each of the following in the form 2^k, where $k \in \mathbb{R}$.

(i) 64 (ii) $\dfrac{\sqrt{32}}{4}$ (iii) $4^x.8^{2x-1}$ (iv) $\sqrt{128^{1-2x}}$.

Solution

(i) $64 = 2\times2\times2\times2\times2\times2 = 2^6$

(ii) $$\frac{\sqrt{32}}{4} = \frac{(2^5)^{\frac{1}{2}}}{2^2} = \frac{2^{\frac{5}{2}}}{2^2} = 2^{\frac{5}{2}-2} = 2^{\frac{1}{2}}$$

(iii) $$4^x.8^{2x-1} = (2^2)^x(2^3)^{2x-1} = 2^{2x}.2^{6x-3} = 2^{8x-3}$$

(iv) $$\sqrt{128^{1-2x}} = \left((2^7)^{1-2x}\right)^{\frac{1}{2}} = \left(2^{7-14x}\right)^{\frac{1}{2}} = 2^{\frac{7}{2}-7x}.$$

Exercises 8.1

Simplify each of the following, without using a calculator:

1. $2^5 \times 2^{-3}$ **2.** $\dfrac{3^5}{(3^{-2})^2}$ **3.** $\dfrac{\sqrt{8}\sqrt{32}}{2^{-3}}$

4. $a^{2x+1}.a^{3x+1}$ **5.** $\dfrac{(a^{x+1})^4}{a^{2-x}}$ **6.** $\sqrt{\dfrac{b^{3x-1}}{b^{x+5}}}$

7. $\left(a^2b^3\right)^{-3}$ **8.** $\dfrac{xy^3}{(x^2y)^{-1}}$ **9.** $\left(\dfrac{p^2q}{q^4}\right)^2$

10. Evaluate $5(4^{3n+1}) - 20(8^{2n})$.

11. If $f(n) = 4(2^n)$, show that $f(n+k) = 2^k f(n)$.

12. Express $2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}}$ in the form 2^k, without using a calculator.

8.2 Equations with the Unknown in the Index

1. Single Power Equal to a Single Power

To solve an equation such as

$$2^{2x-1}=\frac{1}{4^{x+3}}$$

(i) write each side with the same base,

(ii) put the indices equal to each other, and so solve for *x*.

Example 1

Solve for $x\in\mathbb{R}$, $\quad 8^{1-2x}=\sqrt{\frac{1}{32^x}}$.

Solution

(Write each side as a single power with a base of 2.)

$$(2^3)^{1-2x}=(32^{-x})^{\frac{1}{2}}$$
$$2^{3-6x}=\left((2^5)^{-x}\right)^{\frac{1}{2}}$$
$$2^{3-6x}=(2^{-5x})^{\frac{1}{2}}$$
$$2^{3-6x}=2^{-\frac{5x}{2}} \qquad \textit{... now each side is a power with the same base}$$
$$3-6x=-\frac{5x}{2} \qquad \textit{... equating indices}$$
$$6-12x=-5x$$
$$6=7x$$
$$x=\frac{6}{7}$$

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2. Equations which become Quadratic

* An equation such as
$$2^{2x+3}-33(2^x)+4=0$$
can be converted to a quadratic equation by letting *y* be the basic power present, i.e.
$$y=2^x.$$

* To write 2^{2x+3} in terms of $y=2^x$:
$$2^{2x+3}=2^{2x}.2^3$$
$$=(2^x)^2(8)$$
$$=8y^2$$

* We can now write the given equation as
$$8y^2-33y+4=0.$$

* We now solve this equation for y. Don't forget to finish by calculating the corresponding values of x.

Example 2
Solve the equation

$$3^{2x+5} - 4(3^{x+2}) + 1 = 0$$

for $x \in \mathbb{R}$.

Solution

Let $y = 3^x$. Then

(i) $3^{x+2} = 3^x.3^2 = 9(3^x) = 9y$

(ii) $3^{2x+5} = 3^{2x}.3^5 = 243(3^x)^2 = 243y^2$

Then the equation can be written

$$243y^2 - 4(9y) + 1 = 0$$
$$243y^2 - 36y + 1 = 0$$
$$(27y - 1)(9y - 1) = 0$$
$$27y - 1 = 0 \quad \text{or} \quad 9y - 1 = 0$$
$$y = \frac{1}{27} \quad \text{or} \quad y = \frac{1}{9}$$

Then $3^x = \frac{1}{3^3}$ or $3^x = \frac{1}{3^2}$

$$3^x = 3^{-3} \quad \text{or} \quad 3^x = 3^{-2}$$
$$x = -3 \quad \text{or} \quad x = -2.$$

Exercises 8.2

Solve the following equations for $x \in \mathbb{R}$.

1. $2^x = 32$

2. $3^x = \frac{1}{27}$

3. $2^{3x-1} = \frac{1}{64}$

4. $3^{2x+1} = \sqrt{27}$

5. $9^{x+2} = \frac{1}{27^{2x+5}}$

6. $4^{2x-1} = \sqrt{\frac{2}{8^{x+3}}}$

7. By letting $y = 2^x$ and converting to a quadratic equation, solve

$$2^{2x} - 20(2^x) + 64 = 0, \qquad \text{for } x \in \mathbb{R}.$$

8. By letting $y = 3^x$ and converting to a quadratic equation, solve

$$3^{2x+1} - 28(3^x) + 9 = 0, \qquad \text{for } x \in \mathbb{R}.$$

8.3 Logarithms

1. Definition of Logarithm (Log)

Logarithm, or log, is another term for index. For example, in the index statement

$$2^3 = 8,$$

the log is 3.

2. Index Statement and Log Statement

* The index statement

$$2^3 = 8$$

can be written as a log statement in the form

$$\log_2 8 = 3,$$

i.e. the base, 2, is written as a subscript and the value of the power, 8, is written directly after the log.

* The meaning of the symbol

$$\log_a x$$

is the index that must be placed above the base *a* to give the value *x*.

Definition of Logs

$$a^y = x \quad \Leftrightarrow \quad \log_a x = y$$

3. Evaluating a Log

To evaluate a log, without using a calculator:

(i) let the log be *x*,
(ii) write the log statement in index form,
(iii) solve this equation for *x*.

Example 1

Evaluate, without using a calculator, $\log_3 \sqrt{27}$.

Solution

Let $\log_3 \sqrt{27} = x$.

Then $3^x = \sqrt{27}$

$$3^x = \left(3^3\right)^{\frac{1}{2}}$$

$$3^x = 3^{\frac{3}{2}}$$

$$x = \frac{3}{2}.$$

4. Laws of Logs

Laws of Logs

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a x^q = q\log_a x$

4. $\log_a a = 1$ and $\log_a 1 = 0$

5. $\log_a x = \frac{\log_b x}{\log_b a}$ **(Change of Base Law)**

The Laws of Logs are related to the Laws of Indices and can be derived from these laws. These laws are often used in conjunction with each other.

Example 2

If $x = \log_{10} 2$ and $y = \log_{10} 3$, express in terms of x and y:

(i) $\log_{10}\sqrt{\frac{32}{27}}$, (ii) $\log_{10}\frac{4000}{9}$.

Solution

(i)
$$\begin{aligned}
\log_{10}\sqrt{\frac{32}{27}} &= \log_{10}\left(\frac{32}{27}\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\log_{10}\frac{32}{27} && \text{... } \textit{Law 3} \\
&= \frac{1}{2}(\log_{10}32 - \log_{10}27) && \text{... } \textit{Law 2} \\
&= \frac{1}{2}\left(\log_{10}2^5 - \log_{10}3^3\right) \\
&= \frac{1}{2}(5\log_{10}2 - 3\log_{10}3) && \text{... } \textit{Law 3} \\
&= \frac{1}{2}(5x - 3y)
\end{aligned}$$

(ii)
$$\begin{aligned}
\log_{10}\frac{4000}{9} &= \log_{10}4000 - \log_{10}9 && \text{... } \textit{Law 2} \\
&= \log_{10}(1000\times 4) - \log_{10}3^2 \\
&= \log_{10}10^3 + \log_{10}2^2 - \log_{10}3^2 && \text{... } \textit{Law 1} \\
&= 3\log_{10}10 + 2\log_{10}2 - 2\log_{10}3 && \text{... } \textit{Law 3} \\
&= 3 + 2x - 2y. && \text{... } \textit{Law 4}
\end{aligned}$$

Example 3

(i) Show that $\log_a b = \dfrac{1}{\log_b a}$.

(ii) Show that $\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x} + \dfrac{1}{\log_c x} = \log_x(abc)$.

Solution

(i) By Law 5, the Change of Base Law,

$$\log_a b = \frac{\log_b b}{\log_b a}$$

$$= \frac{1}{\log_b a} \qquad \textit{... Law 4}$$

(ii) By part (i),

$$\frac{1}{\log_a x} = \log_x a\,, \quad \frac{1}{\log_b x} = \log_x b, \quad \frac{1}{\log_c x} = \log_x c$$

Then

$$\frac{1}{\log_a x} + \frac{1}{\log_b x} + \frac{1}{\log_c x} = \log_x a + \log_x b + \log_x c$$

$$= \log_x(a \times b \times c)$$

$$= \log_x(abc).$$

Exercises 8.3

Without using a calculator, evaluate each of the following.

1. $\log_4 16$ **2.** $\log_3 \sqrt{27}$ **3.** $\log_4 128$ **4.** $\log_{16} \sqrt{128}$

Write each of the following as a single log.

5. $\log_2 x + \log_2 y$

6. $\log_3 a^2 - \log_3 b$

7. $\log_3 a - 2\log_3 b + 3\log_3 c$

8. $\log_4 x + \frac{1}{2}\log_4 y - 3\log_4 z$

9. (i) If $\log_a x = \log_a y$, show that $x = y$.

(ii) Hence show that if

$$\log_4 x = \log_2 y$$

then $x = y^2$.

10. If $p = \log_{27} q$, express in terms of p

(i) q (ii) $\log_9 q$ (iii) $\log_q 9$.

8.4 Log Equations

1. Definition of a Log Equation
A log equation is an equation containing logs. To solve such equations, we usually need to eliminate the logs from the equation.

2. Solving Log Equations
To eliminate logs from an equation, we can write the equation in one or other of the following two formats.

* **Single log equal to a number:**

$$\log_a X = Y$$

Thus $a^Y = X$

* **Single log equal to a single log to the same base:**

$$\log_a X = \log_a Y$$

Thus $X = Y$.

3. Checking Answers
As long as a is a positive number, $\log_a x$ is **only defined** if $x > 0$. Thus when we solve a log equation, any answer must be checked to make sure that we are not taking the log of a negative number or zero.

Example 1
Solve the equation

$$\log_6(x-3) + \log_6(x+2) = 1,$$

for $x \in \mathbb{R}$.
Solution
(We will write the equation as a single log equal to a number.)

$$\log_6(x-3) + \log_6(x+2) = 1$$
$$\log_6(x-3)(x+2) = 1$$
$$\log_6(x^2 - x - 6) = 1$$
$$x^2 - x - 6 = 6^1$$
$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 4 \quad \text{or} \quad x = -3$$

Check: $x = 4$ OK
$x = -3$ Not OK ($\log_6(-6)$ is not defined.)
Answer: $x = 4$.

Example 2
Solve the equation

$$\log_4(3x+1) = \log_2(x-1),$$

for $x \in \mathbb{R}$.

Solution

(We need to write both sides with the same base, say 2.)

$$\log_4(3x+1) = \frac{\log_2(3x+1)}{\log_2 4} \qquad \textit{... Change of Base Law}$$

$$= \frac{\log_2(3x+1)}{2}$$

Then the equation is

$$\frac{\log_2(3x+1)}{2} = \log_2(x-1)$$

$$\log_2(3x+1) = 2\log_2(x-1)$$

$$\log_2(3x+1) = \log_2(x-1)^2 \qquad \textit{... Law 3}$$

$$3x+1 = x^2 - 2x + 1$$

$$0 = x^2 - 5x$$

$$x(x-5) = 0$$

$$x = 0 \text{ or } x - 5 = 0$$

$$x = 0 \text{ or } x = 5$$

Check: $x = 0$ Not OK (The given equation contains $\log_2(-1)$.)

$x = 5$ OK

Ans: $x = 5$.

4. **Using Logs to Solve Index Equations**

Logs can be used to solve equations with the unknown in the index, if the base and the value cannot be easily expressed as powers with the same base.

Example 3
Find the value of $t \in \mathbb{R}$ if

$$1 \cdot 065^t = 2.$$

Give your answer correct to two decimal places.

Solution

Writing in log form, and using a calculator,

$$1 \cdot 065^t = 2$$

$$t = \log_{1 \cdot 065} 2$$

$$t = 11 \cdot 01, \qquad \textit{... using the } \log_{\square}\square \textit{ function on a calculator}$$

correct to two decimal places.

Exercises 8.4

Solve the following equations for $x \in \mathbb{R}$.

1. $\log_4 x + \log_4 2 = \log_4 28$

2. $\log_3(2x-1) - \log_3(x-4) = 2$

3. $\log_2(x-3) - 2\log_2(x+3) = -5$

4. $\log_2(5x+1) = 2\log_2(x+1)$

5. $\log_3(x+4) = \log_3(x-2) + \log_3(2x-7)$

6. $\log_5(x+3) - \log_5(2x-3) = \log_5(2x+1)$

7. **(i)** Show that $\log_4(5x+1) = \frac{1}{2}\log_2(5x+1)$.

(ii) Hence, or otherwise, solve the equation

$$\log_4(5x+1) = \log_2(x+1)$$

for $x \in \mathbb{R}$.

8. **(i)** Show that $\log_9(4x-15) = \frac{1}{2}\log_3(4x-15)$.

(ii) Hence, or otherwise, solve the equation

$$\log_3(x+3) - \log_9(4x-15) = 1$$

for $x \in \mathbb{R}$.

9. Find the value of $x \in \mathbb{R}$ for which

$$2\cdot 6^x = 9\cdot 47.$$

Give your answer correct to two decimal places.

10. Find the value of $x \in \mathbb{R}$ for which

$$5\cdot 18^x = 167.$$

Give your answer correct to two decimal places.

8.5 Exponential Functions and Relationships

A Exponential Functions and Relationships

1. Definition of an Exponential Function

* An exponential function is a function of the form

$$f : x \to a(b)^x, \qquad \text{where } a,\ b \in \mathbb{R},\ b > 0.$$

* For example, $f(x) = 4(3)^x$ and $f(x) = 1\cdot 86(2\cdot 03)^x$ are both exponential functions.

* There are many real-life examples of exponential functions, e.g. the growth of an investment under compound interest, the decay of a radioactive substance.

2. Exponential Relationship from a Data Table

The following table gives corresponding values for two variables x and y.

x	0	1	2	3	4
y	150	165	$181 \cdot 5$	$199 \cdot 65$	$219 \cdot 615$

* As the changes in the independent variable, x, are fixed, we can check if there is an exponential relationship between x and y by calculating the ratios of successive terms. If these are constant, then there is an exponential relationship between x and y, i.e. y is an exponential function of x.

* Checking,

(i) $\dfrac{165}{150} = 1 \cdot 1$, (ii) $\dfrac{181 \cdot 5}{165} = 1 \cdot 1$, (iii) $\dfrac{199 \cdot 65}{181 \cdot 5} = 1 \cdot 1$, (iv) $\dfrac{219 \cdot 615}{199 \cdot 65} = 1 \cdot 1$.

As these ratios are constant, y is an exponential function of x.

Definition of an Exponential Relationship

The data in a data table represents an exponential relationship between the variables if

(i) for fixed changes in the independent variable,

(ii) the ratios of successive terms is constant.

3. Finding the Equation of an Exponential Relationship from a Data Table

* If there is an exponential relationship between x and y, we can write

$$y = a(b)^x .$$

* We can then use two data pairs from the table to evaluate a and b.

Example 1

The table below gives corresponding values for the variables x and y.

x	0	1	2	3	4
y	150	165	$181 \cdot 5$	$199 \cdot 65$	$219 \cdot 615$

If there is an exponential relationship between x and y, find the values of the constants a and b if

$$y = a(b)^x .$$

Solution

[1] $x = 0$ when $y = 150$.

$$150 = a(b)^0$$

$$150 = a \qquad \text{... as } b^0 = 1$$

[2] $x = 1$ when $y = 165$.

$$165 = 150(b)^1$$
$$165 = 150b$$
$$b = 1\cdot 1$$

Thus $y = 150(1\cdot 1)^x$.

(Note that *b*, the base of the exponential function, is always the same as the constant ratio, as long as the differences in the *x* values is 1.)

B Exponential Problems

1. Exponential Change in Real Life

A quantity *y* changes exponentially with respect to another quantity *x* if the rate of change of *y* is proportional to the current value of *x*. For example,

* each hour, a biological sample increases by 5% of its size at the beginning of that hour,
* each year, a quantity of radioactive substance decrease by 1% of the quantity present at the beginning of that year.

2. Exponential Growth

* If a quantity *y* **increases** exponentially as another quantity *x* increases, then we can write

  $y = a(b)^x$, where $b > 1$.
* The base of the power being **greater than 1** is the key to exponential growth. In practical problems, it is taken that $a > 0$.

3. Exponential Decay

* If a quantity *y* **decreases** exponentially as another quantity *x* increases, then we can write

  $y = a(b)^x$, where $b < 1$.
* The base of the power being **less than 1** is the key to exponential decay. In practical problems, it is taken that $a > 0$.

4. Finding an Index

Logs can be used to evaluate an index. For example, if

$$67\cdot 848 = 2\cdot 1(3\cdot 15)^x$$

then

$$32\cdot 30857 = 3\cdot 15^x$$

and $x = \log_{3\cdot 15} 32\cdot 30857$

$$x = 3\cdot 0289,$$

correct to four decimal places.

Example 2
A biological sample starts with a size of P and grows at the rate of 3% each day after that. Let $f(x)$ represent the size of the sample x days later.
(i) Determine what factor P is multiplied by after a period of one day.
(ii) Express $f(x)$ in terms of x.
(iii) Find, correct to the nearest hour, how long it takes for the sample to double in size.
(iv) Find an expression for x in terms of n if $f(x) = nP$.

Solution
(i) At the end of the first day, the size of the sample is

$$f(1) = P + 3\% \text{ of } P$$
$$f(1) = P + 0\cdot 03P$$
$$f(1) = P(1 + 0\cdot 03)$$
$$f(1) = P(1\cdot 03)$$

Thus the size of the sample is multiplied by the factor $1\cdot 03$ over a period of one day.

(ii) Hence

$$\begin{aligned} f(2) &= f(1)\times(1\cdot 03) \\ &= P(1\cdot 03)\times(1\cdot 03) \\ &= P(1\cdot 03)^2 \end{aligned}$$

and $f(x) = P(1\cdot 03)^x$

(iii) *(We want to find the value of x for which $f(x) = 2P$, i.e. when the sample size is twice its original size.)*

$$f(x) = 2P$$
$$P(1\cdot 03)^x = 2P$$
$$(1\cdot 03)^x = 2$$

Re-writing this equation in log form,

$$x = \log_{1\cdot 03} 2$$

$x = 23\cdot 45$ days ... *using the* $\log_{\square}\square$ *function on a calculator*

$x = 23$ days and 11 hours,
correct to the nearest hour.

(iv) Given

$$f(x) = nP$$
$$P(1\cdot 03)^x = nP$$
$$(1\cdot 03)^x = n$$
$$x = \log_{1\cdot 03} n.$$

Exercises 8.5

1. The table below gives corresponding values for the variables x and y.

x	0	1	2	3	4
y	100	105	$110\cdot25$	$115\cdot7625$	$121\cdot550625$

(i) Use the data in the table to show that there is an exponential relationship between x and y.

(ii) If $y = a(b)^x$,
find the values of the constants a and b.

2. The table below gives corresponding values for the variables x and y.

x	0	2	4	6	8
y	500	480	$460\cdot8$	$442\cdot368$	$424\cdot67328$

(i) Use the data in the table to show that there is an exponential relationship between x and y.

(ii) If $y = a(b)^x$,
find the values of the constants a and b.

3. A laboratory obtains a 10 kg sample of a radioactive substance, which decays over time. The following measurements of weight, W, in grams, were obtained on a yearly basis. Let x represent the number of years from when the substance was obtained.

x / years	0	1	2	3	4
W / grams	10000	9000	8100	7290	6561

(i) Verify that W is an exponential function of x.

(ii) Express W in terms of x.

(iii) Use your expression for W to calculate the weight of radioactive substance present after $2\cdot75$ years.

(iv) Use logs to find the number of years it will take for the amount of radioactive substance to reduce to half the original amount. (This is called the 'half life' of the substance.) Give your answer in years to three decimal places.

(v) Determine the number of years, correct to three decimal places, that it will take for the weight of radioactive substance to reduce to 1 kg.

4. Each year 4% of the quantity of a radioactive substance present decays. The amount of radioactive substance present at the start was 500 grams. Let $y = f(x)$ be the amount of radioactive substance present after x years.

(i) Write down an expression for $f(1)$, the amount present after one year.

(ii) By writing down expressions for $f(2)$ and $f(3)$, find an expression for $f(x)$.

(iii) Find the number of years it takes for the amount of radioactive substance to reduce to 200 grams. Give your answer correct to the nearest year.

5. The number of bacteria present in a culture increases by 3% each hour. The number of bacteria present initially is N.
(i) If $f(t)$ represents the number of bacteria present after t hours, show that
$$f(t) = N(1\cdot 03)^t .$$
(ii) Find the number of hours it takes for the number of bacteria present to increase to $2N$. Give your answer correct to the nearest hour.
(iii) If the number present after 6 hours is calculated to be 143286, find the value of N correct to the nearest unit.

6. A sum of money P, invested in a financial institution, grows by $2\cdot 5\%$ each year after that. Let $A(t)$ be the amount to which the sum has grown after t years.
(i) Find an expression for $A(t)$.
(ii) Find t_1, the value of t for which the sum of money has grown by 50%.
(iii) Investigate if $A(2t_1) = 2P$.
(iv) If $A(t_1 + t_2) = 2P$, find the value of t_2 .

7. A new machine costs €100,000. Its value depreciates by 8% per year after that. Let $f(t)$ be the value of the machine t years after being purchased.
(i) Express $f(t)$ in the form $a(b)^t$.
(ii) Find the number of years, correct to two decimal places, at which the value of the machine is half its original value.
(iii) The company that buys the machine plans on replacing it when it reaches 20% of its original value. Calculate the number of years the machine will have been in use before being replaced.

8.6 Exponential and Log Graphs

A Exponential Graphs

1. Exponential Graphs from a Table of Values and its Properties

* Like other functions, we can construct an exponential graph by forming a table of values.

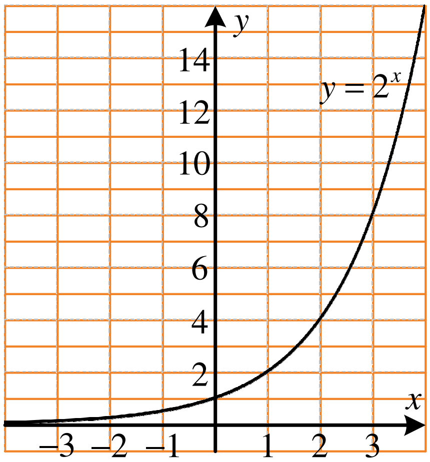
* For example, consider the function
$$f(x) = 2^x$$
and the corresponding graph
$$y = 2^x .$$

* We can construct a table of values for this function.

x	-4	-3	-2	-1	0	1	2	3	4
$y = 2^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

* From this table, we can construct the graph of the curve $y = 2^x$.

* The key features of this curve are as follows.



(i) The curve lies completely above the x-axis.
(ii) On the left, the curve gets closer and closer to the x-axis. The x-axis is called an **asymptote** of the curve. An asymptote of a curve is a line which approximates the curve as the curve tends to infinity.
(iii) On the right, the curve increases more and more rapidly as x increases.

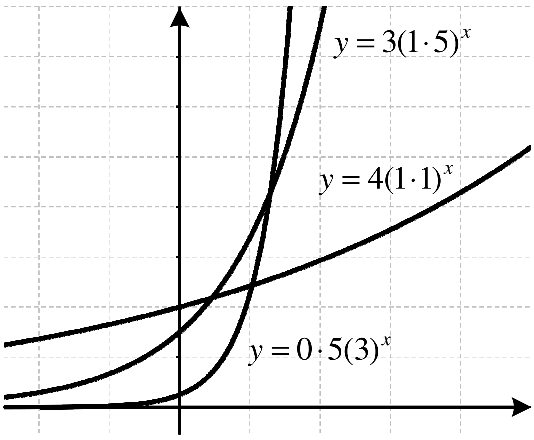
2. Exponential Growth Graphs

* The equation of an exponential growth graph is of the form

$$y = a(b)^x$$

where $a > 0$ and $b > 1$.

* Some of these graphs are shown opposite. Notice that:



(i) The larger the base, b, the more steeply the graph rises on the right.
(ii) The graph intersects the x-axis at the point $(a, 0)$.

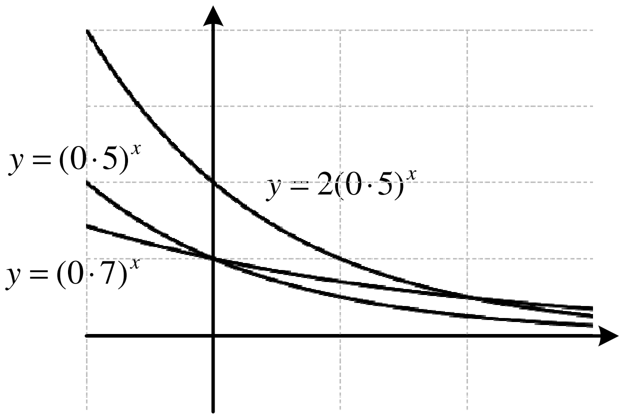
3. Exponential Decay Graphs

* The equation of an exponential decay graph is of the form

$$y = a(b)^x$$

where $a > 0$ and $0 < b < 1$.

* Some of these graphs are shown opposite. Notice that:



(i) The smaller the base, b, the more rapidly the graph reduces on the right.
(ii) The graph interacts the x-axis at the point $(a, 0)$.

B Logarithmic Graphs

1. Exponential and Log Functions are Inverses

* The functions $y = a^x$ and $y = \log_a x$ are inverse functions of each other.
* To demonstrate this, we show that performing one after the other returns us to x.

(i) $\log_a a^x = x \log_a a = x(1) = x$

(ii) To find $a^{\log_a x}$, let

$$y = a^{\log_a x}$$

Then $\log_a y = \log_a x$

$y = x$

Thus $a^{\log_a x} = x$.

2. Graph of Log as Inverse of Exponential

* As discussed above, the functions

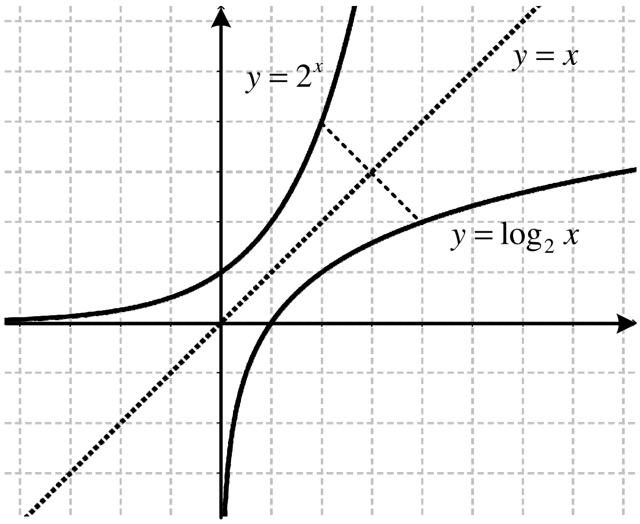
$$f(x) = 2^x$$

and $g(x) = \log_2 x$

are inverses of each other.

* Because of this, the graph of each is the reflection of the other in the line $y = x$.

* The two graphs, $y = 2^x$ and $y = \log_2 x$, are shown opposite.



3. Other Log Graphs

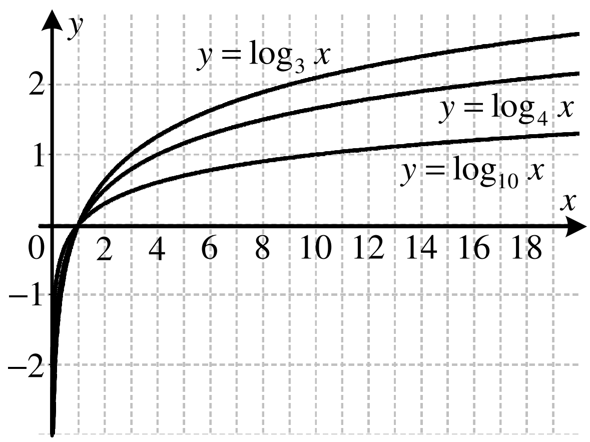
* The diagram opposite shows the three log graphs:

$y = \log_3 x$,

$y = \log_4 x$

and $y = \log_{10} x$.

* Each of these graphs
  (i) is only defined for $x > 0$,
  (ii) crosses the x-axis at $(1, 0)$,
  (iii) tends to $-\infty$ as x tends to 0 from the plus side (the y-axis is an asymptote)
  (iv) has a y co-ordinate of 1 when the x co-ordinate is equal to the base.



4. Constructing Log Graphs

We can construct a log graph by
(i) making a table of values, and
(ii) using our knowledge of the shape of a log graph, as outlined above.

C Solving Equations Using Intersecting Graphs

We can obtain approximate solutions of an equation containing exponential functions or log functions by finding the intersection of the types of graphs we have discussed previously.

Example 1

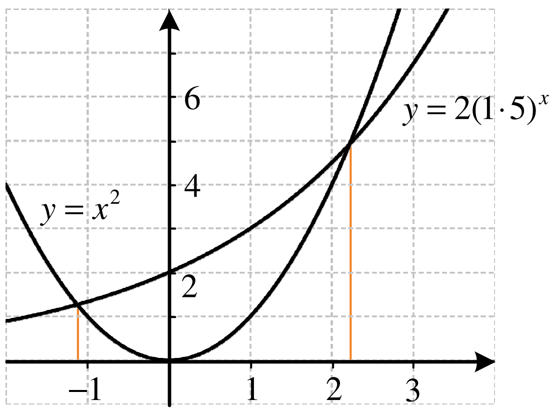
(i) Sketch a rough graph of the function
$$f : x \to 2(1\cdot 5)^x .$$
(ii) Using the same axes and the same scales, sketch a rough graph of the function
$$g : x \to x^2 .$$
(iii) Use your graph to estimate the solutions of the equation
$$2(1\cdot 5)^x = x^2 .$$
(iv) What other methods, if any, could have been used to solve this equation?

Solution

(i) The function is
$$f(x) = 2(1\cdot 5)^x$$
and the corresponding curve is
$$y = 2(1\cdot 5)^x .$$
Constructing a table of values:

x	-1	0	1	2
y	$1\cdot 33$	2	3	$4\cdot 5$

The graph is shown below.



(ii) The function is
$$g(x) = x^2$$
and the corresponding curve is
$$y = x^2 .$$
This curve is also shown on the graph opposite.

(iii) The equation
$$2(1\cdot 5)^x = x^2$$
may be written
$$f(x) = g(x) .$$

The solutions of this equation are the x co-ordinates of the points of intersection of the curves $y = f(x)$ and $y = g(x)$. From the graph, the solutions are $x = -1\cdot 1$ and $x = 2\cdot 2$, correct to one decimal place.

(iv) We have no algebraic method on our course to solve the equation

$$2(1\cdot 5)^x = x^2.$$

One other possible method is to estimate the solutions from tables of values, but the graphical method is undoubtedly the best available.

Exercises 8.6

1. $f : \mathbb{R} \to \mathbb{R}^+ : x \to 3(2)^x$.

(i) By completing the following table of values, construct a graph of the curve $y = f(x)$, for $-3 \le x \le 1$.

x	-3	-2	-1	0	1
$y = f(x)$					

(ii) Using the same axes and the same scales, construct a graph of the function

$$g : x \to x + 3, \quad \text{for } -3 \le x \le 1.$$

(iii) Use your graphs to find the solutions of the equation

$$3(2)^x = x + 3.$$

(iv) What other methods that we have seen can be used to solve this equation? Discuss.

2. $f : \mathbb{R} \to \mathbb{R}^+ : x \to \frac{2}{3}\left(\frac{3}{2}\right)^x$.

(i) By completing the following table of values, construct a graph of the curve $y = f(x)$, for $-2 \le x \le 4$.

x	-2	-1	0	1	2	3	4
$y = f(x)$							

(ii) Using the same axes and the same scales, construct a graph of the function

$$g : x \to \frac{1}{2}x + 1, \quad \text{for } -2 \le x \le 4.$$

(iii) Use your graphs to find the solutions of the equation

$$\frac{2}{3}\left(\frac{3}{2}\right)^x = \frac{1}{2}x + 1.$$

(iv) What other methods that we have seen can be used to solve this equation?

3. Two functions are

$$f:\mathbb{R}\to\mathbb{R}^{+}: x\to 2(2\cdot5)^{x}$$

and $$g:\mathbb{R}\to\mathbb{R}^{+}: x\to 3(0\cdot5)^{x}.$$

(i) By completing the following table of values, construct a graph of the curve $y=f(x)$, for $-2\le x\le 2$.

x	-2	-1	0	1	2
$y=f(x)$					

(ii) Using the same axes and the same scales, sketch a graph of $y=g(x)$, for $-2\le x\le 2$.

(iii) Use your graphs to find the solution of the equation

$$f(x)=g(x).$$

(iv) Check your answer to part (iii) by using algebra.

4. $f:\mathbb{R}^{+}\to\mathbb{R}: x\to\log_5 x$.

(i) Copy and complete the table:

x	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25
$y=f(x)$					

Hence construct the graph $y=f(x)$.

(ii) Using the same axes and the same scales, sketch the graph of

$$g:x\to x-2.$$

(iii) Use your graph to find the solutions of the equation

$$\log_5 x = x-2.$$

(iv) What other methods that we have seen can be used to solve this equation? Discuss.

8.7 Natural Exponential and Natural Log

1. The Number '*e*'

* The number *e* is a famous and important mathematical constant. Its value is approximately

$$e=2\cdot718.$$

* The origin of this number will be seen in a later chapter.

2. Natural Exponential Function

* The natural exponential function, or sometimes simply 'the' exponential function, is

$$f(x)=e^{x},$$

where *e* is approximately $2\cdot718$.

* The natural exponential function obeys all the usual Laws of Indices, e.g.
  (i) $e^{2x}.e^3 = e^{2x+3}$
  (ii) $(e^x)^2 = e^{2x}$.

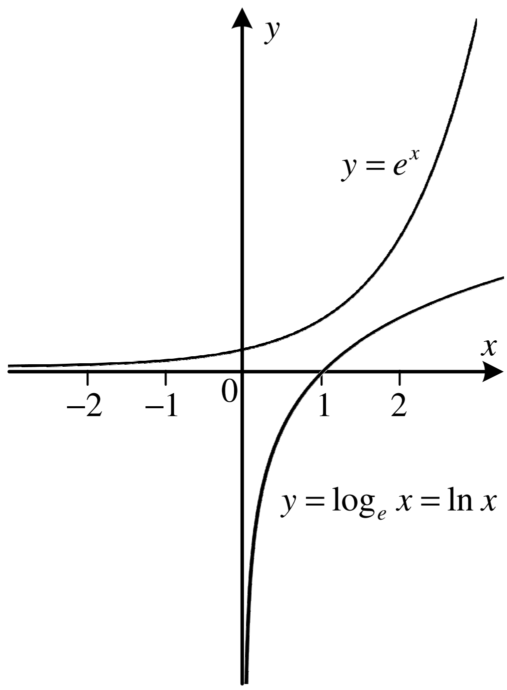
3. Natural Log Function

* The natural log function is
  $$f(x) = \log_e x$$
  where e is approximately $2\cdot718$.
* $\log_e$ can also be written 'ln'. Thus we can also write
  $$f(x) = \ln x.$$
* The natural log function obeys all the usual Laws of Logs, e.g.
  (i) $\ln(x.y) = \ln x + \ln y$
  (ii) $\ln x^4 = 4\ln x$.

4. Natural Exponential and Natural Log are Inverses

Like other exponential and log functions with the same base, the natural exponential and natural log functions are inverses. Thus

(i) $\ln e^x = x$
(ii) $e^{\ln x} = x$.



5. Graphs

* As inverse functions, the graphs
  $$y = e^x$$
  and $y = \ln x$
  are reflections of each other in the line $y = x$.

* These graphs are shown opposite.

6. Other Exponential and Log Graphs

* Like any exponential or log graphs, we can plot a natural exponential graph, e.g.
  $$y = 3e^{2x},$$
  or a natural log graph, e.g.
  $$y = 2\ln(x+3),$$
  by making a table of values.

* Note that $y = e^{2x}$ rises more rapidly than $y = e^x$, for $x > 0$, and that $y = e^{-2x}$ will decrease more rapidly than $y = e^{-x}$, for $x > 0$.

Example 1
Show that

$$\ln\left(xe^{2x}\right) = 2x + \ln x.$$

Solution

$$\begin{aligned}\ln\left(xe^{2x}\right) &= \ln x + \ln e^{2x} \\ &= \ln x + 2x \ln e \\ &= 2x + \ln x.\end{aligned}$$

7. Exponential Growth

In scientific work, it is very common to see a quantity, $Q(t)$ which grows exponentially with time, *t*, being expressed in the form

$$Q(t) = Ae^{bt},$$

where b is a positive constant.

8. Exponential Decay

Also in scientific work, it is very common to see a quantity, $Q(t)$, which decays exponentially with time, *t*, being expressed in the form

$$Q(t) = Ae^{-bt},$$

where b is a positive constant.

Example 2
In a laboratory experiment, a quantity, $Q(t)$, of a chemical was observed at various points in time, *t*. Time is measured in minutes from the initial observation. The table below gives the results.

t	0	1	2	3	4
$Q(t)$	$5\cdot9000$	$4\cdot8305$	$3\cdot9549$	$3\cdot2380$	$2\cdot6510$

Q follows a rule of the form $Q(t) = Ae^{-bt}$, where A and b are constants.

(i) Use two of the observations from your table to evaluate A and b. Verify your values by taking another observation from the table.

(ii) Hence find the value of Q after 10 minutes, i.e. $Q(10)$.

(iii) Estimate the time that it takes for the quantity of the chemical to reduce to 10% of the original amount. Give your answer in minutes, correct to three decimal places.

(iv) It is suggested that another model for Q is $Q(t) = Ap^t$. Express p in terms of b, correct to five decimal places.

Solution

(i) $Q(0)=5\cdot 9$: $Ae^{-b(0)}=5\cdot 9$

$Ae^{0}=5\cdot 9$

$A=5\cdot 9$

$Q(1)=4\cdot 8305$: $5\cdot 9e^{-b(1)}=4\cdot 8305$

$e^{-b}=\dfrac{4\cdot 8305}{5\cdot 9}$

$e^{-b}=0\cdot 8187288$

$-b=\ln 0\cdot 8187288$

$-b=-0\cdot 2$

$b=0\cdot 2$

Thus $Q(t)=5\cdot 9e^{-0\cdot 2t}$

To check:

$$Q(4)=5\cdot 9e^{-0\cdot 2(4)}=5\cdot 9e^{-0\cdot 8}=2\cdot 6510$$

Correct.

(ii) $Q(10)=5\cdot 9e^{-0\cdot 2(10)}$

$=5\cdot 9e^{-2}$

$=0\cdot 7895$

(iii) 10% of the original amount is $0\cdot 59$. Thus

$Q(t)=0\cdot 59$

$5\cdot 9e^{-0\cdot 2t}=0\cdot 59$

$e^{-0\cdot 2t}=0\cdot 1$

$-0\cdot 2t=\ln 0\cdot 1$

$-0\cdot 2t=-2\cdot 302585$

$t=11\cdot 513$ minutes.

(iv) $Q(t)=5\cdot 9e^{-0\cdot 2t}$

$Q(t)=5\cdot 9\left(e^{-0\cdot 2}\right)^{t}$

$Q(t)=5\cdot 9(0\cdot 81873)^{t}$

Thus $p=0\cdot 81873$, correct to five decimal places.

Exercises 8.7

Simplify each of the following expressions.

1. $\log_e\left(x^2e^{-x}\right)$ **2.** $\log_e e^{\sin x+2}$ **3.** $\log_e\left(x^4e^{\cos x}\right)$

4. $\ln(x^2\sqrt{x+1})$ **5.** $\ln(x^3e^{\sin x})$ **6.** $\ln\left(\dfrac{\sqrt{x}}{e^{4x}}\right)$

7. A chemical reaction starts with 1000 grams of reactant X, which then reduces exponentially as the reaction proceeds. The amount of X, in grams, present after t minutes is given by

$$A(t) = Ce^{-bt}, \quad \text{where } C \text{ and } b \text{ are constants.}$$

The following table gives some values of A.

t	0	1	2	3	4
$A(t)$	1000				$367 \cdot 88$

(i) Find the value of C and the value of b. Hence complete the table above.
(ii) Find the value of t for which $A(t) = 50$.
(iii) Another reactant, Y, also starts with 1000 grams and reduces exponentially with time, so that the amount of Y remaining, $B(t)$, is given by

$$B(t) = Ce^{-dt}.$$

If after 4 minutes, there is less of Y remaining than of X, suggest a possible value for the constant d.

8. The amount a company owes on a loan grows exponentially. The amount owed after t years, $A(t)$, is given by the following table.

t	0	1	2	3	4
$A(t)$		$5724 \cdot 46$	$5958 \cdot 08$		

A is given by a rule of the form $A(t) = Pe^{bt}$, where P and b are constants.
(i) Find the value of P and the value of b, and hence complete the table.
(ii) Find the number of years for the amount owed to double from its initial amount.
(iii) If $A(t)$ can also be written $A(t) = P(1+i)^t$, find the value of the constant i.

9. A bacteria colony grows by 4% per day.
(i) If the size at time $t = 0$ is 200, express $Q(t)$, the size after t days, in the form $Q(t) = Ae^{bt}$, where A and b are constants.
(ii) Find, correct to two decimal places, the time it takes for the colony to double in size.
(iii) Show that $\dfrac{Q(t+k)}{Q(t)}$ is independent of t.

10. In 2000, 100 grams of radium, a radioactive substance, were stored. This quantity decays exponentially over time. Let $Q(t) = Ae^{-bt}$ represent the amount of radium remaining t years later.
(i) The half-life of radium is 1602 years, i.e. it takes 1602 years for half of the original amount to decay. Find the values of the constants A and b.
(ii) Find the mass of radium that will remain in the year 4000, correct to two decimal places.
(iii) Find the mass of radium that will decay between the years 3000 and 4000.

8.8 Scientific Notation

1. Definition of Scientific Notation

* A number is said to be written in **scientific notation**, or in **standard form**, when it is in the form $a \times 10^n$, where $1 \le a < 10$ and $n \in \mathbb{Z}$, the set of integers.
* This form is generally used for very large or very small numbers.
* For example,
  (i) $2 \cdot 3 \times 10^{14}$ and
  (ii) $5 \cdot 7 \times 10^{-11}$
  are both numbers written in scientific notation.

Scientific Notation
$a \times 10^n$, where $1 \le a < 10$ and $n \in \mathbb{Z}$.

2. Converting from Decimal Form to Scientific Notation

To write a number in scientific notation, we shift the decimal point to the right or the left, to get a number between 1 and 10, and multiply by the appropriate power of 10. For example,

(i) $382400 = 3 \cdot 824 \times 10^5$

move 5 places to the left

(ii) $0 \cdot 0003176 = 3 \cdot 176 \times 10^{-4}$

move 4 places to the right

3. Converting from Scientific Notation to Decimal Form

To convert a number from scientific notation, $a \times 10^n$, to ordinary decimal form, move the decimal point *n* spaces, to the right if *n* is positive and to the left if *n* is negative. Add zeros before or zeros behind as necessary. For example,

(i) $7 \cdot 8 \times 10^8 = 780000000$

move 8 places to the right

(ii) $6 \cdot 27 \times 10^{-6} = 0 \cdot 00000627$

move 6 places to the left

4. Entering Numbers in Scientific Notation on a Calculator

(a) Casio Natural Display Calculators

To enter $4\cdot 57\times 10^{-7}$, press the following keys:

$\boxed{4}\ \boxed{\cdot}\ \boxed{5}\ \boxed{7}\ \boxed{\times 10^x}\ \boxed{(-)}\ \boxed{7}$

(b) Sharp Write View Calculators

To enter $4\cdot 57\times 10^{-7}$, press the following keys:

$\boxed{4}\ \boxed{.}\ \boxed{5}\ \boxed{7}\ \boxed{\text{Exp}}\ \boxed{(-)}\ \boxed{7}$

5. Adding and Subtracting Numbers in Scientific Notation

* Numbers given in scientific notation can be added or subtracted on a calculator. The answer may be given in decimal form, and if the answer is required in scientific notation, we will have to convert it manually.

* To add or subtract the numbers without a calculator, we can either convert them into decimal form, or write each with the same power of 10. This is so that we can take out the power of 10 as a common factor. For example,

$$(1\cdot 6\times 10^4)+(3\cdot 9\times 10^5)=(0\cdot 16\times 10^5)+(3\cdot 9\times 10^5)$$ ... *one place to the left, thus we change* 10^4 *to* 10^5

$$=(0\cdot 16+3\cdot 9)\times 10^5$$
$$=4\cdot 06\times 10^5$$

6. Multiplying and Dividing Numbers in Scientific Notation

To multiply or divide two numbers in scientific notation, without using a calculator, deal with the as and the 10^n s separately. It may be necessary to adjust the answer to leave it in scientific notation. For example,

$$(3\cdot 2\times 10^6)\times(6\cdot 8\times 10^4)=(3\cdot 2\times 6\cdot 8)\times(10^6\times 10^4)$$
$$=21\cdot 76\times 10^{10}$$
$$=2\cdot 176\times 10^{11}$$ ... *moving the decimal point one place to the left changes* 10^{10} *to* 10^{11}

Example 1

Without using a calculator, express

$$\frac{(3\cdot 34\times 10^5)+(2\cdot 6\times 10^4)}{7\cdot 2\times 10^{-3}}$$

in the form $a\times 10^n$, where $1\le a<10$ and $n\in\mathbb{Z}$.

Solution

$$\begin{aligned}\frac{(3\cdot 34\times 10^{5})+(2\cdot 6\times 10^{4})}{7\cdot 2\times 10^{-3}} &= \frac{(3\cdot 34\times 10^{5})+(0\cdot 26\times 10^{5})}{7\cdot 2\times 10^{-3}}\\ &= \frac{(3\cdot 34+0\cdot 26)\times 10^{5}}{7\cdot 2\times 10^{-3}}\\ &= \frac{3\cdot 6\times 10^{5}}{7\cdot 2\times 10^{-3}}\\ &= \frac{3\cdot 6}{7\cdot 2}\times 10^{5-(-3)}\\ &= 0\cdot 5\times 10^{8}\\ &= 5\times 10^{7}\end{aligned}$$

Example 2

A quantity of a radioactive substance starts with $2\cdot 4\times 10^{24}$ atoms. Each month after that 45% of the remaining substance decays. Write in the form $a\times 10^{n}$, where $1\le a<10$ and $n\in\mathbb{Z}$, the number of radioactive atoms remaining after one year.

Solution

Let $P(t)$ be the number of atoms remaining after *t* months. Then

$$\begin{aligned}P(t) &= (2\cdot 4\times 10^{24})(1-0\cdot 45)^{t}\\ &= (2\cdot 4\times 10^{24})(0\cdot 55)^{t}\end{aligned}$$

After one year, $t=12$, and

$$\begin{aligned}P(12) &= (2\cdot 4\times 10^{24})\times(0\cdot 55)^{12}\\ &= (2\cdot 4\times 10^{24})\times(7\cdot 662\times 10^{-4})\\ &= (2\cdot 4\times 7\cdot 662)\times 10^{24-4}\\ &= 18\cdot 3888\times 10^{20}\\ &= 1\cdot 83888\times 10^{21}\end{aligned}$$

Exercises 8.8

Express each of the following in the form $a\times 10^{n}$, where $1\le a<10$ and $n\in\mathbb{Z}$. You may use your calculator.

1. 56000000	**2.** 1830000	**3.** 6185000
4. 351200000	**5.** 120100	**6.** 8915000000
7. $0\cdot 000045$	**8.** $0\cdot 001825$	**9.** $0\cdot 00000067$
10. $0\cdot 0009156$	**11.** $0\cdot 008325$	**12.** $0\cdot 01567$

Without using a calculator, write each of the following in scientific notation and in decimal form. You should check your answer by using your calculator.

13. $(5\cdot4\times10^4)+(8\cdot7\times10^3)$

14. $(8\cdot2\times10^7)+(1\cdot73\times10^8)$

15. $(9\cdot31\times10^{-5})-(8\cdot25\times10^{-6})$

16. $(2\cdot83\times10^4)-(8\cdot72\times10^3)$

17. $(2\cdot8\times10^5)\times(1\cdot23\times10^6)$

18. $(5\cdot2\times10^{-4})\times(1\cdot5\times10^7)$

19. $\dfrac{(3\cdot34\times10^5)+(2\cdot6\times10^4)}{7\cdot2\times10^{-3}}$

20. $\dfrac{(3\cdot25\times10^4)+(5\cdot23\times10^3)}{3\cdot773\times10^{-5}}$

21. $\dfrac{(5\cdot2\times10^5)+(5\times10^4)}{2\cdot85\times10^8}$

22. $\dfrac{(1\cdot27\times10^5)+(7\cdot03\times10^5)}{3\cdot32\times10^{-3}}$

23. The distance from earth to the Andromeda galaxy is approximately 2 million light years, and a light year is approximately $9\cdot47\times10^{12}$ km.
(i) Express the distance from earth to Andromeda in kilometres in scientific notation.
(ii) In the future, a new spaceship is capable of travelling at 4×10^8 km/h. How long would it take this spaceship to reach Andromeda from earth?

24. The country of Bananistan has a national debt of 127 billion euro (1 billion is equal to 1000 million). The population of Bananistan is $5\cdot4$ million.
(i) Calculate the average amount owed by every citizen of Bananistan.
(ii) If the working population of Bananistan is $3\cdot74$ million, calculate the amount owed by every working person in Bananistan.

25. The number of cells in a bacteria colony is initially estimated to be $5\cdot4\times10^9$. Each hour after that the number of cells grows by 6%. Estimate the number of cells in the colony after
(i) 1 hour,
(ii) 1 day.
Give your answers in scientific notation.

26. A manufacturing company has a number of machines which produce widgets. One machine produces $2\cdot5\times10^6$ widgets in a day, two other machines each produce $3\cdot7\times10^6$ widgets in a day, and one super machine produces $1\cdot2\times10^7$ widgets in a day.
(i) Calculate the number of widgets produced by the company in a day.
(ii) If 4% of the widgets fail a quality test, calculate the number of good widgets produced by the company in a five day week.

27. A large company invests $3\cdot5$ billion euro in research and development in 2012. It intends to increase this amount by 8% each year after that. Write in scientific notation the amount it intends to invest in research and development between 2012 and 2015 inclusive.

Revision Exercises 8

1. Solve the equation

$$2^x.2^{x+1} = 10$$

giving your answer
(i) in log form
(ii) correct to three decimal places.

2. Show that

$$\frac{8^n \times 2^{2n}}{4^{3n}} = \frac{1}{2^n}.$$

3. (i) Solve the equation

$$x + \frac{1}{x} = \frac{10}{3}.$$

(ii) Hence solve the equation

$$e^y + e^{-y} = \frac{10}{3}.$$

4. Solve the equation

$$2^{x+1} = 3^x,$$

giving your answer correct to three decimal places.

5. Siobhan is given a dose of radioactive medicine, which decays at the rate of 15% per hour after that. If the original dose is of 80 mg, and $f(x)$ mg represents the amount left in her body x hours after receiving the dose,
(i) calculate the amount left after 4 hours,
(ii) find the value of x when $f(x) = 20$,
(iii) the amount by which $f(x)$ decreases in the third hour.

6. $f(x) = Pe^{-kx}$ gives the amount of radioactive substance present x years after it starts to decay.
(i) What is the initial amount present, i.e. $f(0)$?
(ii) If it takes 10000 years for half the amount of radioactive substance to decay, express the value of k in scientific notation.
(iii) Express $f(1000)$ in terms of P.

7. A company wants to replace an existing machine in five years time. It estimates that the replacement will cost €500000 in five years time. It decides to set aside an amount €P now so that it will grow at the rate of 4% per year and will provide the finance to purchase the replacement. Calculate the value of P.

8. A bacteria colony grows at the rate of 8% per day.
(i) How many days will it take the colony to double in size?
(ii) How many days will it take the colony to treble in size?
(iii) If initially there are 8×10^7 bacteria present, write in scientific notation the number of bacteria present after 30 days.

9. A bacteria colony starts with a size of A and grows exponentially. It doubles in size in 10 minutes. Write the size, $Q(t)$, of the colony after t minutes in the form

$$Q(t) = Ae^{bt}$$

giving the value of b, correct to four decimal places.

Solutions to Exercises

Exercises 8.1

1. $2^5 \times 2^{-3} = 2^{5-3} = 2^2 = 4$

2. $\dfrac{3^5}{(3^{-2})^2} = \dfrac{3^5}{3^{-4}} = 3^{5-(-4)} = 3^9$

3. $\dfrac{\sqrt{8}\sqrt{32}}{2^{-3}} = \dfrac{\sqrt{256}}{2^{-3}} = \dfrac{16}{2^{-3}} = \dfrac{2^4}{2^{-3}} = 2^{4-(-3)} = 2^7 = 128$

4. $a^{2x+1}.a^{3x+1} = a^{(2x+1)+(3x+1)} = a^{5x+2}$

5. $\dfrac{(a^{x+1})^4}{a^{2-x}} = \dfrac{a^{4x+4}}{a^{2-x}} = a^{(4x+4)-(2-x)} = a^{5x+2}$

6. $$\begin{aligned}\sqrt{\frac{b^{3x-1}}{b^{x+5}}} &= \sqrt{b^{(3x-1)-(x+5)}} \\ &= \sqrt{b^{2x-6}} \\ &= \left(b^{2x-6}\right)^{\frac{1}{2}} \\ &= b^{x-3}\end{aligned}$$

7. $(a^2b^3)^{-3} = (a^2)^{-3}(b^3)^{-3} = a^{-6}b^{-9}$

8. $\dfrac{xy^3}{(x^2y)^{-1}} = \dfrac{xy^3}{x^{-2}y^{-1}} = x^{1-(-2)}y^{3-(-1)} = x^3y^4$

9. $\left(\dfrac{p^2q}{q^4}\right)^2 = \left(\dfrac{p^2}{q^3}\right)^2 = \dfrac{(p^2)^2}{(q^3)^2} = \dfrac{p^4}{q^6}$

10. $$\begin{aligned}5(4^{3n+1}) - 20(8^{2n}) &= 5\left((2^2)^{3n+1}\right) - 20\left((2^3)^{2n}\right) \\ &= 5\left(2^{6n+2}\right) - 20\left(2^{6n}\right) \\ &= 5\left(2^{6n}.2^2\right) - 20\left(2^{6n}\right) \\ &= 20(2^{6n}) - 20(2^{6n}) \\ &= 0\end{aligned}$$

11. $$\begin{aligned}f(n) &= 4(2^n) \\ f(n+k) &= 4(2^{n+k}) \\ &= 4(2^n.2^k) \\ &= 2^k\left[4(2^n)\right] \\ &= 2^k f(n)\end{aligned}$$

12. $2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} = 4\left(2^{\frac{1}{4}}\right) = 2^2.2^{\frac{1}{4}} = 2^{2+\frac{1}{4}} = 2^{\frac{9}{4}}$.

Exercises 8.2

1. $2^x = 32$
$2^x = 2^5$
$x = 5$

2. $3^x = \frac{1}{27}$
$3^x = \frac{1}{3^3}$
$3^x = 3^{-3}$
$x = -3$

3. $2^{3x-1} = \frac{1}{64}$
$2^{3x-1} = \frac{1}{2^6}$
$2^{3x-1} = 2^{-6}$
$3x - 1 = -6$
$3x = -5$
$x = -\frac{5}{3}$

4. $3^{2x+1} = \sqrt{27}$
$3^{2x+1} = (3^3)^{\frac{1}{2}}$
$3^{2x+1} = 3^{\frac{3}{2}}$
$2x + 1 = \frac{3}{2}$
$2x = \frac{1}{2}$
$x = \frac{1}{4}$

5. $9^{x+2} = \frac{1}{27^{2x+5}}$
$(3^2)^{x+2} = \frac{1}{(3^3)^{2x+5}}$
$3^{2x+4} = \frac{1}{3^{6x+15}}$
$3^{2x+4} = 3^{-6x-15}$
$2x + 4 = -6x - 15$
$8x = -19$
$x = -\frac{19}{8}$

6. $4^{2x-1} = \sqrt{\frac{2}{(2^3)^{x+3}}}$

$$(2^2)^{2x-1} = \sqrt{\frac{2}{2^{3x+9}}}$$
$$2^{4x-2} = \sqrt{2^{1-(3x+9)}}$$
$$2^{4x-2} = (2^{-3x-8})^{\frac{1}{2}}$$
$$2^{4x-2} = 2^{\frac{1}{2}(-3x-8)}$$
$$4x-2 = \frac{1}{2}(-3x-8)$$
$$8x-4 = -3x-8$$
$$11x = -4$$
$$x = -\frac{4}{11}$$

7. Let $y = 2^x$. Then

$2^{2x} = (2^x)^2 = y^2$

The equation is

$$y^2 - 20y + 64 = 0$$
$$(y-4)(y-16) = 0$$
$$y = 4 \quad \text{or} \quad y = 16$$
$$2^x = 2^2 \quad \text{or} \quad 2^x = 2^4$$
$$x = 2 \quad \text{or} \quad x = 4$$

8. Let $y = 3^x$. Then

$3^{2x+1} = 3(3^x)^2 = 3y^2$

The equation is

$$3y^2 - 28y + 9 = 0$$
$$(3y-1)(y-9) = 0$$
$$3y-1 = 0 \quad \text{or} \quad y-9 = 0$$
$$y = \frac{1}{3} \quad \text{or} \quad y = 9$$
$$3^x = 3^{-1} \quad \text{or} \quad 3^x = 3^2$$
$$x = -1 \quad \text{or} \quad x = 2.$$

Exercises 8.3

1. $\log_4 16 = x$

$$4^x = 16$$
$$4^x = 4^2$$
$$x = 2$$

2. $\log_3 \sqrt{27} = x$

$$3^x = (3^3)^{\frac{1}{2}}$$
$$3^x = 3^{\frac{3}{2}}$$
$$x = \frac{3}{2}$$

3. $\log_4 128 = x$

$(2^2)^x = 2^7$

$2^{2x} = 2^7$

$2x = 7$

$x = \frac{7}{2}$

4. $\log_{16} \sqrt{128} = x$

$(2^4)^x = (2^7)^{\frac{1}{2}}$

$2^{4x} = 2^{\frac{7}{2}}$

$4x = \frac{7}{2}$

$x = \frac{7}{8}$

5. $\log_2 x + \log_2 y = \log_2 xy$

6. $\log_3 a^2 - \log_3 b = \log_3 \frac{a^2}{b}$

7. $\log_3 a - 2\log_3 b + 3\log_3 c = \log_3 a - \log_3 b^2 + \log_3 c^3$

$$= \log_3 \frac{ac^3}{b^2}$$

8. $\log_4 x + \frac{1}{2}\log_4 y - 3\log_4 z = \log_4 x + \log_4 y^{\frac{1}{2}} - \log_4 z^3$

$$= \log_4 \frac{x\sqrt{y}}{z^3}$$

9. (i) $\log_a x = \log_a y$

$a^{\log_a y} = x$

$y = x$

(ii) $\log_4 x = \log_2 y$

$\log_4 x = \frac{\log_4 y}{\log_4 2}$

$\log_4 x = \frac{\log_4 y}{\frac{1}{2}} = 2\log_4 y$

$\log_4 x = \log_4 y^2$

$x = y^2$

10. (i) $p = \log_{27} q$

$q = 27^p$

(ii) $\log_9 q = \log_9 27^p = p\log_9 27$

Let $\log_9 27 = x$

$9^x = 27$

$(3^2)^x = 3^3$

$3^{2x} = 3^3$

$2x = 3$

$$x = \frac{3}{2}$$

Then

$$\log_9 q = \frac{3}{2} p$$

(iii) $\log_q 9 = \frac{\log_9 9}{\log_9 q} = \frac{1}{\log_9 q} = \frac{1}{\frac{3}{2} p} = \frac{2}{3p}$.

Exercises 8.4

1. $\log_4 x + \log_4 2 = \log_4 28$

$\log_4 2x = \log_4 28$

$2x = 28$

$x = 14$

2. $\log_3 (2x-1) - \log_3 (x-4) = 2$

$\log_3 \frac{2x-1}{x-4} = 2$

$\frac{2x-1}{x-4} = 3^2 = 9$

$2x - 1 = 9x - 36$

$35 = 7x$

$x = 5$

3. $\log_2 (x-3) - 2\log_2 (x+3) = -5$

$\log_2 (x-3) - \log_2 (x+3)^2 = -5$

$\log_2 \frac{x-3}{(x+3)^2} = -5$

$\frac{x-3}{(x+3)^2} = 2^{-5} = \frac{1}{32}$

$32(x-3) = x^2 + 6x + 9$

$32x - 96 = x^2 + 6x + 9$

$0 = x^2 - 26x + 105$

$(x-5)(x-21) = 0$

$x - 5 = 0$ or $x - 21 = 0$

$x = 5$ or $x = 21$

4. $\log_2 (5x+1) = 2\log_2 (x+1)$

$\log_2 (5x+1) = \log_2 (x+1)^2$

$5x + 1 = x^2 + 2x + 1$

$0 = x^2 - 3x$

$x(x-3) = 0$

$x = 0$ or $x - 3 = 0$

$x = 0$ or $x = 3$

(OK) (OK)

Ans: $x = 0$ or $x = 3$

5. $\log_3(x+4) = \log_3(x-2) + \log_3(2x-7)$

$\log_3(x+4) = \log_3(x-2)(2x-7)$

$x+4 = 2x^2 - 11x + 14$

$0 = 2x^2 - 12x + 10$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x-5=0$ or $x-1=0$

$x=5$ or $x=1$

(OK) (not OK)

Ans: $x=5$

6. $\log_5(x+3) - \log_5(2x-3) = \log_5(2x+1)$

$\log_5 \dfrac{x+3}{2x-3} = \log_5(2x+1)$

$\dfrac{x+3}{2x-3} = 2x+1$

$x+3 = (2x-3)(2x+1)$

$x+3 = 4x^2 - 4x - 3$

$0 = 4x^2 - 5x - 6$

$(x-2)(4x+3) = 0$

$x-2=0$ or $4x+3=0$

$x=2$ or $x = -\dfrac{3}{4}$

(OK) (not OK)

Ans: $x=2$

7. (i) $\log_4(5x+1) = \dfrac{\log_2(5x+1)}{\log_2 4} = \dfrac{1}{2}\log_2(5x+1)$

(ii) $\log_4(5x+1) = \log_2(x+1)$

$\dfrac{1}{2}\log_2(5x+1) = \log_2(x+1)$

$\log_2(5x+1) = 2\log_2(x+1)$

$\log_2(5x+1) = \log_2(x+1)^2$

$5x+1 = x^2 + 2x + 1$

$0 = x^2 - 3x$

$x(x-3) = 0$

$x=0$ or $x=3$

8. (i) $\log_9(4x-15) = \dfrac{\log_3(4x-15)}{\log_3 9} = \dfrac{1}{2}\log_3(4x-15)$

(ii) $\log_3(x+3) - \log_9(4x-15) = 1$

$\log_3(x+3) - \dfrac{1}{2}\log_3(4x-15) = 1$

$2\log_3(x+3) - \log_3(4x-15) = 2$

$\log_3 \dfrac{(x+3)^2}{4x-15} = 2$

$$\frac{x^2+6x+9}{4x-15}=3^2=9$$
$$x^2+6x+9=36x-135$$
$$x^2-30x+144=0$$
$$(x-6)(x-24)=0$$
$$x-6=0 \quad \text{or} \quad x-24=0$$
$$x=6 \quad \text{or} \quad x=24$$

9. $2\cdot6^x=9\cdot47$

$x=\log_{2\cdot6}9\cdot47$

$x=2\cdot35$

10. $5\cdot18^x=167$

$x=\log_{5\cdot18}167$

$x=3\cdot11$.

Exercises 8.5

1. (i) $\frac{105}{100}=1\cdot05,\ \frac{110\cdot25}{105}=1\cdot05,\ \frac{115\cdot7625}{110\cdot25}=1\cdot05,\ \frac{121\cdot550625}{115\cdot7625}=1\cdot05$

As these ratios are constant, y is an exponential function of x.

(ii) $y=a(b)^x$

$x=0,\ y=100:\quad 100=a(b)^0$

$a=100$

$x=1,\ y=105:\quad 105=100(b)^1$

$b=1\cdot05$

Thus $y=100(1\cdot05)^x$.

2. (i) $\frac{480}{500}=0\cdot96,\ \frac{460\cdot8}{480}=0\cdot96,\ \frac{442\cdot368}{460\cdot8}=0\cdot96,\ \frac{424\cdot67328}{442\cdot368}=0\cdot96$

As these ratios are constant, y is an exponential function of x.

(ii) $y=a(b)^x$

$x=0,\ y=500:\quad 500=a(b)^0$

$a=500$

$x=2,\ y=480:\quad 480=500(b)^2$

$b^2=0\cdot96$

$b=0\cdot9798$

Thus $y=500(0\cdot9798)^x$.

3. (i) Calculating ratios of successive terms:

$\frac{9000}{10000}=0\cdot9,\ \frac{8100}{9000}=0\cdot9,\ \frac{7290}{8100}=0\cdot9,\ \frac{6561}{7290}=0\cdot9$

As the ratio is constant, W is an exponential function of x.

(ii) Let $W=a(b)^x$.

Then $b=0\cdot9$, and

$10000=a(0\cdot9)^0$

$a=10000$

Thus

$W = 10000(0 \cdot 9)^x$

(iii) When $x = 2 \cdot 75$,

$W = 10000(0 \cdot 9)^{2 \cdot 75}$

$W = 7484 \cdot 57$ g

(iv) Half the original amount is 5000 g.

$5000 = 10000(0 \cdot 9)^x$

$(0 \cdot 9)^x = 0 \cdot 5$

$x = \log_{0 \cdot 9} 0 \cdot 5$

$x = 6 \cdot 579$

(v) $W = 1000$:

$1000 = 10000(0 \cdot 9)^x$

$(0 \cdot 9)^x = 0 \cdot 1$

$x = \log_{0 \cdot 9} 0 \cdot 1$

$x = 21 \cdot 854$.

4. (i) $f(0) = 500$

$f(1) = f(0) - 4\%$ of $f(0)$

$f(1) = 96\%$ of $f(0)$

$f(1) = f(0) \times 0 \cdot 96 = 500 \times 0 \cdot 96$

(ii) $f(2) = f(1) \times 0 \cdot 96$

$f(2) = [f(0) \times 0 \cdot 96] \times 0 \cdot 96 = f(0) \times (0 \cdot 96)^2$

$f(3) = f(0) \times (0 \cdot 96)^3$

$f(x) = f(0) \times (0 \cdot 96)^x = 500(0 \cdot 96)^x$

(iii) $200 = 500(0 \cdot 96)^x$

$0 \cdot 96^x = 0 \cdot 4$

$x = \log_{0 \cdot 96} 0 \cdot 4$

$x = 22$, correct to the nearest year.

5. (i) $f(0) = N$

$f(1) = N + 3\%$ of N

$f(1) = N(1 \cdot 03)$

$f(2) = N(1 \cdot 03)^2$

$f(t) = N(1 \cdot 03)^t$

(ii) $2N = N(1 \cdot 03)^t$

$1 \cdot 03^t = 2$

$t = \log_{1 \cdot 03} 2$

$t = 23$, correct to the nearest hour

(iii) $143286 = N(1 \cdot 03)^6$

$$N = \frac{143286}{1 \cdot 03^6}$$

$N = 12000$.

6. **(i)** $A(t) = P(1 + 0 \cdot 025)^t$

$A(t) = P(1 \cdot 025)^t$

(ii) $A(t_1) = 1 \cdot 5P$

$P(1\cdot025)^{t_1} = 1\cdot5P$

$(1\cdot025)^{t_1} = 1\cdot5$

$t_1 = \log_{1\cdot025} 1\cdot5$

$t_1 = 16\cdot42$

(iii) $A(2t_1) = A(32\cdot84)$

$= P(1\cdot025)^{32\cdot84}$

$= 2\cdot25P$

$\neq 2P$

(iv) $A(16\cdot42 + t_2) = 2P$

$P(1\cdot025)^{16\cdot42+t_2} = 2P$

$1\cdot025^{16\cdot42+t_2} = 2$

$16\cdot42 + t_2 = \log_{1\cdot025} 2$

$t_2 = 11\cdot65$.

7. **(i)** $f(t) = 100000(1 - 0\cdot08)^t$

$f(t) = 100000(0\cdot92)^t$

(ii) $f(t) = 50000$

$100000(0\cdot92)^t = 50000$

$(0\cdot92)^t = 0\cdot5$

$t = \log_{0\cdot92} 0\cdot5$

$t = 8\cdot31$, correct to two decimal places

(iii) $f(t) = 20000$

$100000(0\cdot92)^t = 20000$

$(0\cdot92)^t = 0\cdot2$

$t = \log_{0\cdot92} 0\cdot2$

$t = 19\cdot3$ years.

Exercises 8.6

1. **(i)** $f(x) = 3(2)^x$

Table:

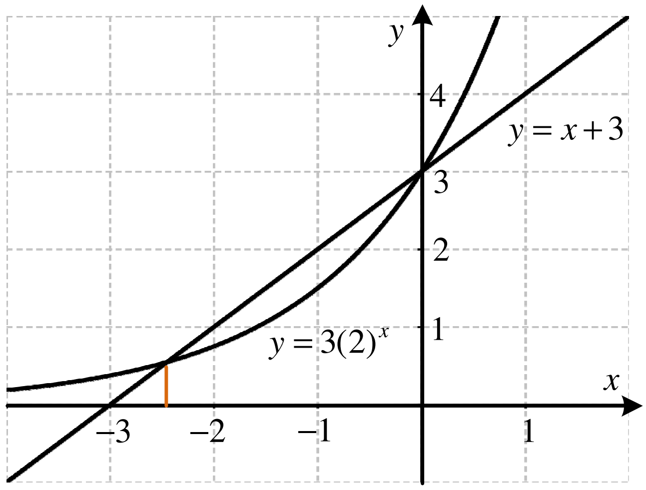
x	-3	-2	-1	0	1
$y = f(x)$	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	3	6

The graph is constructed below.

(ii) $g(x) = x + 3$

The linear graph $y = x + 3$ contains the points $(-3, 0)$ and $(0, 3)$, and is shown opposite.

(iii) From the graph, the solutions are approximately $-2\cdot5$ and 0.



(iv) We could compare values in the tables of values, but there is no algebraic method we have seen that can be used to solve this equation.

2. **(i)** $f(x)=\frac{2}{3}\left(\frac{3}{2}\right)^x$

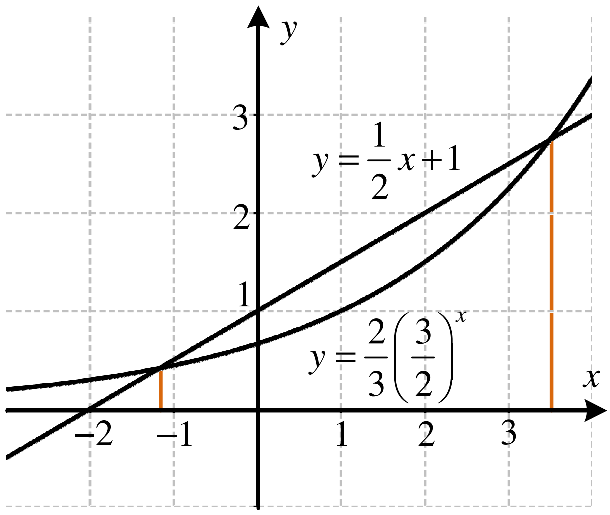
Table:

x	-2	-1	0	1	2	3	4
$y=f(x)$	$0\cdot3$	$0\cdot44$	$0\cdot67$	1	$1\cdot5$	$2\cdot25$	$3\cdot375$

The graph is constructed below.

(ii) $g(x)=\frac{1}{2}x+1$

The linear graph $y=\frac{1}{2}x+1$ contains the points $(0,1)$ and $(2,2)$, and is shown opposite.



(iii) From the graph, the solutions of the equation are approximately $-1\cdot2$ and $3\cdot4$.

(iv) We could compare values in the tables of values, but there is no algebraic method we have seen that can be used to solve this equation.

3. **(i)** $f(x)=2(2\cdot5)^x$

Table:

x	-2	-1	0	1	2
$y=f(x)$	$0\cdot32$	$0\cdot8$	2	5	$12\cdot5$

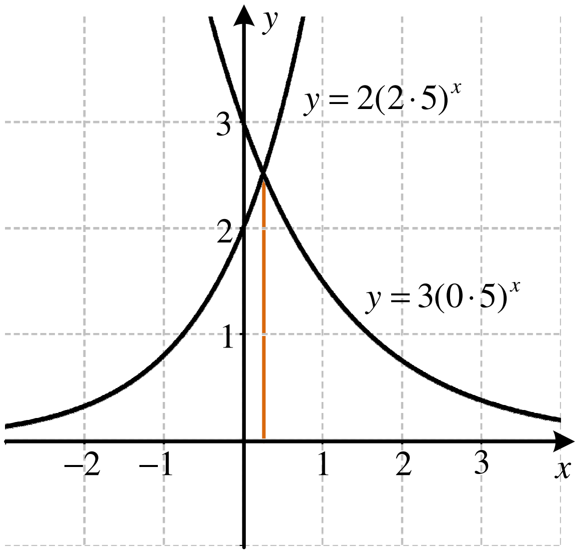
The graph is constructed below.

(ii) $g(x)=3(0\cdot5)^x$

Table:

x	-2	-1	0	1	2
$y=g(x)$	12	6	3	$1\cdot5$	$0\cdot75$

The graph is constructed below.



(iii) From the graph, the only solution of the equation is $0\cdot25$.

(iv) $f(x)=g(x)$

$$2(2\cdot5)^x=3(0\cdot5)^x$$

$$2\left(\frac{5}{2}\right)^x=3\left(\frac{1}{2}\right)^x$$

$$2\times\frac{5^x}{2^x}=3\times\frac{1}{2^x}$$

$$2\times5^x=3$$

$$5^x=1\cdot5$$

$$x=\log_5 1\cdot5$$

$$x=0\cdot2519$$

4. **(i)** $f(x)=\log_5 x$

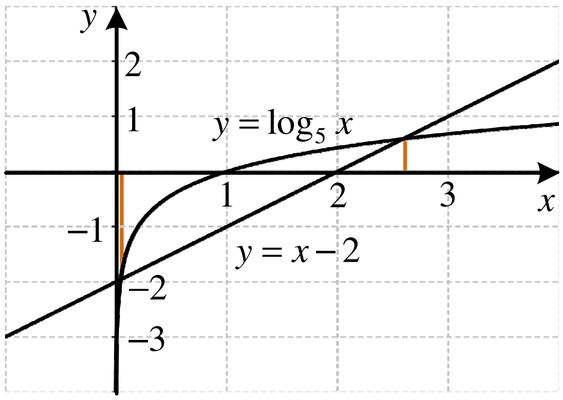
Table:

x	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25
$y=f(x)$	-2	-1	0	1	2

The graph is constructed below.

(ii) $g(x)=x-2$

The linear graph $y=x-2$ contains the points $(2,0)$ and $(0,-2)$, and is shown opposite.



(iii) From the graph, the solutions of the equation are approximately $0\cdot1$ and $2\cdot7$.

(iv) We could compare values in the tables of values, but there is no algebraic method we have seen that can be used to solve this equation.

Exercises 8.7

1. $\log_e(x^2e^{-x}) = \log_e x^2 + \log_e e^{-x}$
$= 2\log_e x + (-x)\log_e e$
$= 2\log_e x - x$

2. $\log_e e^{\sin x+2} = (\sin x + 2)\log_e e$
$= \sin x + 2$

3. $\log_e(x^4e^{\cos x}) = \log_e x^4 + \log_e e^{\cos x}$
$= 4\log_e x + (\cos x)\log_e e$
$= 4\log_e x + \cos x$

4. $\ln(x^2\sqrt{x+1}) = \ln x^2 + \ln(x+1)^{\frac{1}{2}}$
$= 2\ln x + \frac{1}{2}\ln(x+1)$

5. $\ln(x^3e^{\sin x}) = \ln x^3 + \ln e^{\sin x}$
$= 3\ln x + \sin x(\ln e)$
$= 3\ln x + \sin x$

6. $\ln\left(\frac{\sqrt{x}}{e^{4x}}\right) = \ln\sqrt{x} - \ln e^{4x}$
$= \ln x^{\frac{1}{2}} - (4x)\ln e$
$= \frac{1}{2}\ln x - 4x$

7. (i) $A(t) = Ce^{-bt}$

$A(0) = 1000$: $1000 = Ce^0$
$C = 1000$

$A(4) = 367\cdot 88$: $367\cdot 88 = 1000e^{-b(4)}$
$0\cdot 36788 = e^{-4b}$
$-4b = \ln 0\cdot 36788$
$-4b = -0\cdot 9999$
$b = 0\cdot 25$

Thus $A(t) = 1000e^{-0\cdot 25t}$

and $A(1) = 1000e^{-0\cdot 25} = 778\cdot 8$
$A(2) = 1000e^{-0\cdot 5} = 606\cdot 53$
$A(3) = 1000e^{-0\cdot 75} = 472\cdot 37$

(ii) $A(t) = 50$
$1000e^{-0\cdot 25t} = 50$
$e^{-0\cdot 25t} = 0\cdot 05$
$-0\cdot 25t = \ln 0\cdot 05$
$-0\cdot 25t = -2\cdot 9957$
$t = 11\cdot 98$

(iii) $d > 0\cdot 25$, e.g. $d = 0\cdot 5$.

8. (i) $A(t) = Pe^{bt}$

$A(1) = 5724 \cdot 46: \quad 5724 \cdot 64 = Pe^{b}$ **... 1**

$A(2) = 5958 \cdot 08: \quad 5958 \cdot 08 = Pe^{2b}$ **... 2**

Dividing **2** by **1**,

$$\frac{Pe^{2b}}{Pe^{b}} = \frac{5958 \cdot 08}{5724 \cdot 64}$$

$e^{b} = 1 \cdot 04077811$

$b = \ln 1 \cdot 04077811$

$b = 0 \cdot 03997$

1: $5724 \cdot 64 = P(1 \cdot 04077811)$

$P = 5500 \cdot 35$

Thus $A(t) = 5500 \cdot 35 e^{0 \cdot 03997t}$

and $A(0) = P = 5500 \cdot 35$

$A(3) = Pe^{3b} = 6201 \cdot 04$

$A(4) = Pe^{4b} = 6453 \cdot 91$

(ii) $A(t) = 2P$

$2P = Pe^{0 \cdot 03997t}$

$2 = e^{0 \cdot 03997t}$

$0 \cdot 03997t = \ln 2$

$0 \cdot 03997t = 0 \cdot 6931471806$

$t = 17 \cdot 34$

(iii) $A(t) = P(1+i)^{t}$

$(1+i)^{t} = e^{bt}$

$1+i = e^{b}$

$1+i = 1 \cdot 04077811$

$i = 0 \cdot 04077811$

9. **(i)** $Q(t) = 200(1 \cdot 04)^{t}$

Let $e^{b} = 1 \cdot 04$

$b = \ln 1 \cdot 04 = 0 \cdot 03922071315$

Thus $Q(t) = 200 e^{0 \cdot 03922071315t}$

(ii) $Q(t) = 400$

$200(1 \cdot 04)^{t} = 400$

$(1 \cdot 04)^{t} = 2$

$t = \log_{1 \cdot 04} 2 = 17 \cdot 67$

(iii) $\dfrac{Q(t+k)}{Q(t)} = \dfrac{200(1 \cdot 04)^{t+k}}{200(1 \cdot 04)^{t}} = 1 \cdot 04^{k}$

which is independent of *t*.

10. **(i)** $Q(t) = Ae^{-bt}$

$Q(1602) = \dfrac{1}{2}A$

$Ae^{-1602b} = \dfrac{1}{2}A$

$e^{-1602b} = 0 \cdot 5$

$-1602b = \ln 0\cdot 5$
$-1602b = -0\cdot 6931471806$
$b = 0\cdot 000432676$
and $A = 100$.

(ii) In 4000, $t = 2000$.
$Q(2000) = 100e^{-0\cdot 000432676(2000)}$
$Q(2000) = 42\cdot 09$ grams

(iii) In 3000, $t = 1000$.
$Q(1000) = 100e^{-0\cdot 000432676(1000)}$
$Q(1000) = 64\cdot 88$
The mass of radium that decreases between 3000 and 4000 is
$64\cdot 88 - 42\cdot 09$
$= 22\cdot 79$ grams.

Exercises 8.8

1. $56000000 = 5\cdot 6\times 10^7$
2. $1830000 = 1\cdot 83\times 10^6$
3. $6185000 = 6\cdot 185\times 10^6$
4. $351200000 = 3\cdot 512\times 10^8$
5. $121000 = 1\cdot 21\times 10^5$
6. $8915000000 = 8\cdot 915\times 10^9$
7. $0\cdot 000045 = 4\cdot 5\times 10^{-5}$
8. $0\cdot 001825 = 1\cdot 825\times 10^{-3}$
9. $0\cdot 00000067 = 6\cdot 7\times 10^{-7}$
10. $0\cdot 0009156 = 9\cdot 156\times 10^{-4}$
11. $0\cdot 008325 = 8\cdot 325\times 10^{-3}$
12. $0\cdot 01567 = 1\cdot 567\times 10^{-2}$
13. $(5\cdot 4\times 10^4) + (8\cdot 7\times 10^3) = (5\cdot 4\times 10^4) + (0\cdot 87\times 10^4)$
$= 6\cdot 27\times 10^4$
$= 62700$
14. $(8\cdot 2\times 10^7) + (1\cdot 73\times 10^8) = (0\cdot 82\times 10^8) + (1\cdot 73\times 10^8)$
$= 2\cdot 55\times 10^8$
$= 255000000$
15. $(9\cdot 31\times 10^{-5}) - (8\cdot 25\times 10^{-6}) = (9\cdot 31\times 10^{-5}) - (0\cdot 825\times 10^{-5})$
$= 8\cdot 485\times 10^{-5}$
$= 0\cdot 00008485$
16. $(2\cdot 83\times 10^4) - (8\cdot 72\times 10^3) = (2\cdot 83\times 10^4) - (0\cdot 872\times 10^4)$
$= 1\cdot 958\times 10^4$
$= 19580$
17. $(2\cdot 8\times 10^5)\times(1\cdot 23\times 10^6) = 3\cdot 444\times 10^{11}$
$= 344400000000$
18. $(5\cdot 2\times 10^{-4})\times(1\cdot 5\times 10^7) = 7\cdot 8\times 10^3$
$= 7800$

19. $$\frac{(3\cdot34\times10^5)+(2\cdot6\times10^4)}{7\cdot2\times10^{-3}}=\frac{3\cdot6\times10^5}{7\cdot2\times10^{-3}}$$
$$=0\cdot5\times10^8$$
$$=5\times10^7$$
$$=50000000$$

20. $$\frac{(3\cdot25\times10^4)+(5\cdot23\times10^3)}{3\cdot773\times10^{-5}}=\frac{3\cdot773\times10^4}{3\cdot773\times10^{-5}}$$
$$=10^9$$
$$=100000000$$

21. $$\frac{(5\cdot2\times10^5)+(5\times10^4)}{2\cdot85\times10^8}=\frac{5\cdot7\times10^5}{2\cdot85\times10^8}$$
$$=2\times10^{-3}$$
$$=0\cdot002$$

22. $$\frac{(1\cdot27\times10^5)+(7\cdot03\times10^5)}{3\cdot32\times10^{-3}}=\frac{8\cdot3\times10^5}{3\cdot32\times10^{-3}}$$
$$=2\cdot5\times10^8$$
$$=250000000$$

23. **(i)** distance $=(2000000)\times(9\cdot47\times10^{12})$ km
$=1\cdot894\times10^{19}$ km

(ii) Time $=\dfrac{\text{distance}}{\text{speed}}$
$$=\frac{1\cdot894\times10^{19}}{4\times10^8}$$
$=4\cdot735\times10^{10}$ hours
$=5401551$ years ... *assuming* $365\cdot25$ *days per year*

24. **(i)** Average amount owed $=\dfrac{\text{total debt}}{\text{population}}$
$$=\frac{127\times1000000000}{5\cdot4\times1000000}$$
$$=\frac{1\cdot27\times10^{11}}{5\cdot4\times10^6}$$
$=€23519$

(ii) Average amount owed per working person
$$=\frac{1\cdot27\times10^{11}}{3\cdot74\times10^6}$$
$=€33957$

25. **(i)** No. of cells $=(5\cdot4\times10^9)\times(1\cdot06)$
$=5\cdot724\times10^9$

(ii) No. of cells $=\left(5\cdot4\times10^9\right)\times(1\cdot06)^{24}$
$=2\cdot186\times10^{10}$

26. **(i)** In one day, number of widgets
$=(2\cdot5\times10^6)+2(3\cdot7\times10^6)+(1\cdot2\times10^7)$
$=2\cdot19\times10^7$
$=21900000$

(ii) Number of good widgets in one day

$= (0 \cdot 96) \times \left(2 \cdot 19 \times 10^7\right)$

$= 21024000$

Number of good widgets in one five day week

$= 105120000$

$= 1 \cdot 0512 \times 10^8$

27. 2012: $3 \cdot 5 \times 10^9$

2013: $(3 \cdot 5 \times 10^9) \times 1 \cdot 08 = 3 \cdot 78 \times 10^9$

2014: $(3 \cdot 5 \times 10^9) \times (1 \cdot 08)^2 = 4 \cdot 0824 \times 10^9$

2015: $(3 \cdot 5 \times 10^9) \times (1 \cdot 08)^3 = 4 \cdot 409 \times 10^9$

Total investment

$= (3 \cdot 5 + 3 \cdot 78 + 4 \cdot 0824 + 4 \cdot 409) \times 10^9$

$= 15 \cdot 7714 \times 10^9$

$= 1 \cdot 57717 \times 10^{10}$ euros

$= €15 \cdot 77$ billion.

Revision Exercises 8

1. (i) $2^x . 2^{x+1} = 10$

$2^{2x+1} = 10$

$\log_2 10 = 2x + 1$

$2x = \log_2 10 - 1$

$x = \frac{1}{2}\left(\log_2 10 - 1\right)$

(ii) $x = 1 \cdot 161$

2. $\frac{8^n \times 2^{2n}}{4^{3n}} = \frac{(2^3)^n \times 2^{2n}}{(2^2)^{3n}}$

$= \frac{2^{5n}}{2^{6n}}$

$= \frac{1}{2^n}$

3. (i) $x + \frac{1}{x} = \frac{10}{3}$ Multiply by $3x$.

$3x^2 + 3 = 10x$

$3x^2 - 10x + 3 = 0$

$(3x - 1)(x - 3) = 0$

$x = \frac{1}{3}$ or $x = 3$

(ii) $e^y + e^{-y} = \frac{10}{3}$

$e^y + \frac{1}{e^y} = \frac{10}{3}$

From part (i),

$$e^y = \frac{1}{3} \quad \text{or} \quad e^y = 3$$
$$y = \ln\frac{1}{3} \quad \text{or} \quad y = \ln 3$$

4. $2^{2x+1} = 3^x$

$2^x.2^1 = 3^x$

$2 = \dfrac{3^x}{2^x}$

$2 = \left(\dfrac{3}{2}\right)^x$

$2 = 1\cdot 5^x$

$x = \log_{1\cdot 5} 2$

$x = 1\cdot 710$

5. **(i)** $f(x) = 80(1-0\cdot 15)^x = 80(0\cdot 85)^x$

$f(4) = 80(0\cdot 85)^4$

$= 41\cdot 76 \text{ mg}$

(ii) $f(x) = 20$

$80(0\cdot 85)^x = 20$

$(0\cdot 85)^x = 0\cdot 25$

$x = \log_{0\cdot 85} 0\cdot 25$

$x = 8\cdot 53$

(iii) The third hour is from $x = 2$ to $x = 3$.

$f(2) = 80(0\cdot 85)^2 = 57\cdot 8$

$f(3) = 80(0\cdot 85)^3 = 49\cdot 13$

Reduction in third hour

$= f(2) - f(3)$

$= 57\cdot 8 - 49\cdot 13$

$= 8\cdot 67$ mg.

6. **(i)** $f(x) = Pe^{-kx}$

$f(0) = Pe^0 = P$

(ii) $f(10000) = 0\cdot 5P$

$Pe^{-10000k} = 0\cdot 5P$

$e^{-10000k} = 0\cdot 5$

$-10000k = \ln 0\cdot 5$

$-10000k = -0\cdot 6931$

$k = 0\cdot 00006931$

$k = 6\cdot 931\times 10^{-5}$

(iii) $f(1000) = Pe^{-(6\cdot 931\times 10^{-5})\times 10^3}$

$f(1000) = Pe^{-0\cdot 06931}$

$f(1000) = 0\cdot 9330P$.

7. €P invested now at 4% per annum compound interest in 5 years time will be worth

€$P(1\cdot 04)^5 =$ €$1\cdot 21665P$

If this is to pay for the replacement machine, then

$1\cdot 21665P = 500000$

$$P = \frac{500000}{1 \cdot 21665}$$
$$P = 410965$$

8. **(i)** Let P be the initial size of the colony and $f(x)$ be the size of the colony after x days. Then

$$f(x) = P(1 \cdot 08)^x$$

To double in size:

$$f(x) = 2P$$
$$P(1 \cdot 08)^x = 2P$$
$$(1 \cdot 08)^x = 2$$
$$x = \log_{1 \cdot 08} 2$$
$$x = 9 \cdot 01$$

Thus it will take $9 \cdot 01$ days to double in size.

(ii) To treble in size:

$$f(x) = 3P$$
$$P(1 \cdot 08)^x = 3P$$
$$(1 \cdot 08)^x = 3$$
$$x = \log_{1 \cdot 08} 3$$
$$x = 14 \cdot 27$$

Thus it will take $14 \cdot 27$ days to treble in size.

(iii) $P = 8 \times 10^7$ and

$$f(30) = [8 \times 10^7] \times (1 \cdot 08)^{30}$$
$$f(30) = [8 \times 10^7] \times 10 \cdot 063$$
$$f(30) = 8 \cdot 05 \times 10^8 .$$

9. $Q(t) = Ae^{bt}$

$$Q(10) = 2A$$
$$Ae^{10b} = 2A$$
$$e^{10b} = 2$$
$$10b = \ln 2$$
$$10b = 0 \cdot 6931$$
$$b = 0 \cdot 06931 .$$