Name:

THE INSTITUTE OF EDUCATION

MATHEMATICS

Hilary Dorgan

Leaving Certificate

Higher Level

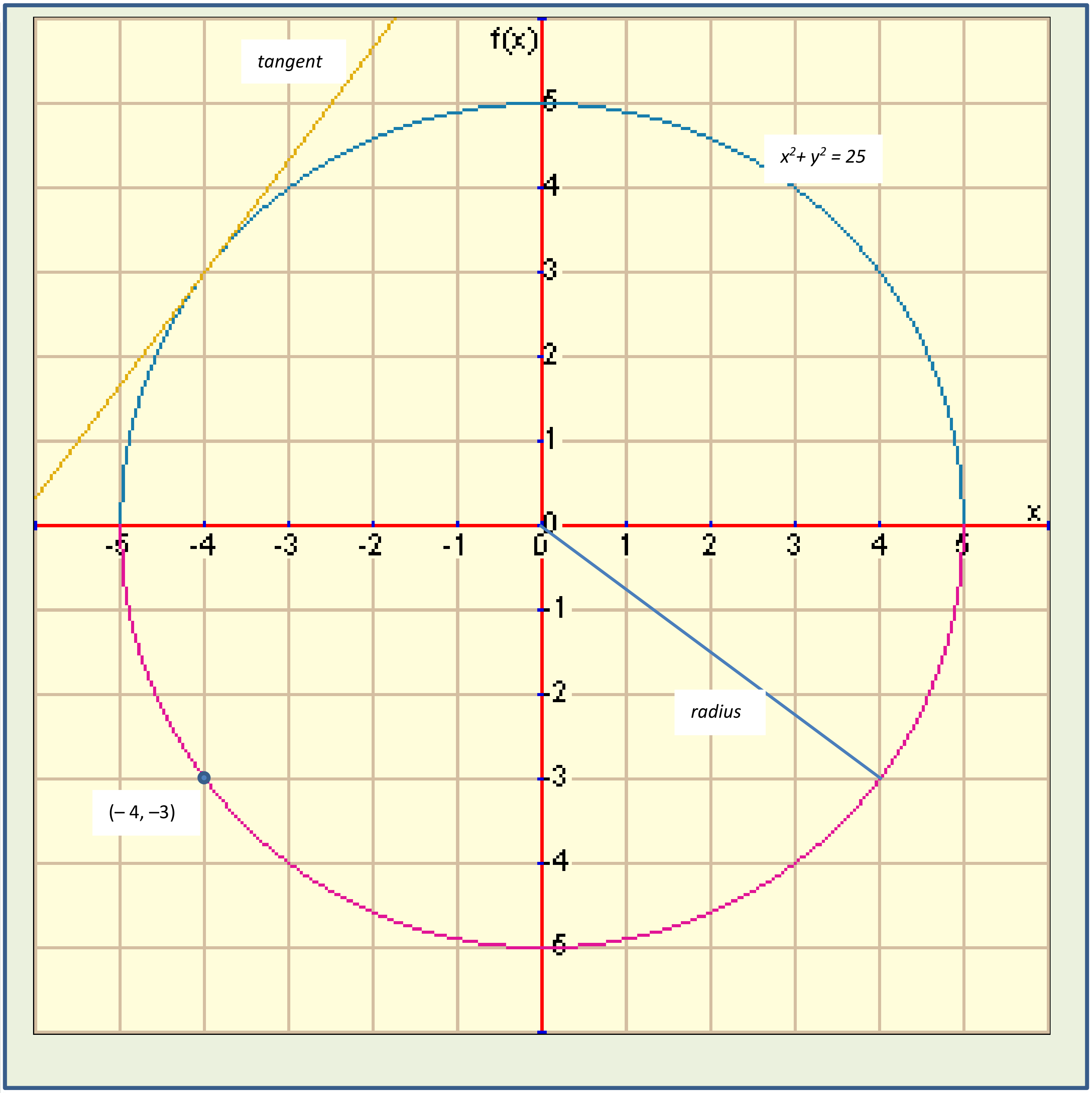
2021 - 2022

CIRCLE

NOTES

1

THE INSTITUTE OF EDUCATION FOUNDED 1969


f(x)
tangent
$x^2 + y^2 = 25$
radius
(– 4, –3)
x
-5
-4
-3
-2
-1
0
1
2
3
4
5

CONTENTS

CIRCLE SYLLABUS

- Recognise that $(x-h)^2 + (y-k)^2 = r^2$ represents the relationship between the x and y coordinates of points on a circle of centre (h, k) and radius r. (OL)
- Solve problems involving a line and a circle with centre $(0, 0)$. (OL)
- Recognize that $x^2 + y^2 + 2gx + 2fy + c = 0$ represents the relationship between the x and y coordinates of points on a circle of centre $(-g, -f)$ and radius r where $r = \sqrt{g^2 + f^2 - c}$. (HL)
- Solve problems involving a line and a circle. (HL)

WHAT YOU MUST KNOW

Recognize the equation of a circle in various forms (e.g. degree 2, cannot have an xy term, coefficients of x^2 and y^2equal)

Familiarity with all the formulae for coordinate geometry of the Straight Line

Find the centre and radius of a circle given its equation in any form

Find the equation of a circle given its centre and radius

Find the equation of a circle given the two end points of a diameter

Check whether a particular point is inside, outside or on a given circle

Familiarity with Junior Cert theorems involving circles

Find the point of intersection between a line and a circle

Prove that a given line is a tangent to a circle

Find the equation of a tangent of a circle given the equation of the circle and their point of intersection

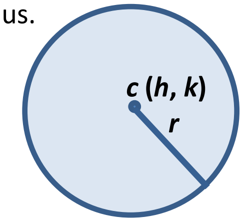
Find the equation of a circle given its centre and given the equation of a tangent to the circle

Find the equations of the two tangents that can be drawn from a given point outside a given circle

Solve problems involving g , f and c.

Equation of a Circle

<u>**What is a Circle**</u>: A circle is a **set of points** that are **equidistant from a common point** ***c*** **(*h, k*)** called the **centre**. The fixed distance ***r*** from the centre to any point on the circle is called the radius.



The standard equation of a circle with centre ***c*** **(*h, k*)** and radius ***r*** is: $(x-h)^2 + (y-k)^2 = r^2$.
If the centre of the circle is at the origin, ***c*** **(0, 0),** this equation becomes $x^2 + y^2 = r^2$.

Example 1: Find the equation of a circle whose centre is at $(3, -7)$ and has a radius of 2.

Solution 1: Given $(h, k) = (3, -7)$ and $r = 2$

$\Rightarrow$ required circle equation: $(x-3)^2 + (y-(-7))^2 = 2^2$

$\Rightarrow$ required equation: $(x-3)^2 + (y+7)^2 = 4$

Example 2: Find the equation of a circle that has a diameter with the endpoints given by the points $\boldsymbol{a}(-3, 4)$ and $\boldsymbol{b}(5, 10)$.

Solution 2: Whenever we want the equation of a circle, we look always for two pieces of information
(i) What is the ***centre*** of the circle?
(ii) What is the ***radius*** of the circle?

The ***centre*** of this circle, ***c***, is the midpoint of the line segment or diameter ***ab***.

$c = \left(\frac{-3+5}{2}, \frac{4+10}{2}\right) = (1, 7)$ [Using the mid-point formula]

The ***radius*** of this circle is half the length of the diameter between ***a*** and ***b***.

$$r = \frac{1}{2}\sqrt{(5+3)^2 + (10-4)^2} = \frac{1}{2}\sqrt{64+36} = \frac{1}{2} \times 10 = 5$$

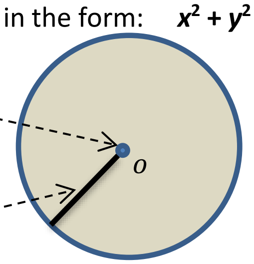
Using standard equation $(x-h)^2 + (y-k)^2 = r^2 \Rightarrow (x-1)^2 + (y-7)^2 = 5^2$

$\Rightarrow (x-1)^2 + (y-7)^2 = 25$

The equation of a circle is also very often given in the form: $x^2 + y^2 + 2gx + 2fy + c = 0$.

The centre (o) of this circle is: $(-g, -f)$

and the length of the radius of this circle is:



[Both formulae on *page 19* of Tables]

$$r = \sqrt{g^2 + f^2 - c}$$

<u>**Note**</u>: The equation of a circle should always have the following properties:
(i) it is of degree 2, (ii) it has no xy term (iii) the coefficients of x^2 and y^2 are equal

Example 3: Find the centre and radius of the circle with equation $x^2 + y^2 - 4x - 6y + 9 = 0$.

Solution 3: We will do this question by two different methods. [This will verify the previous formula given.]

Method 1: This method involves re-writing the given equation in more standard form with all ***x*** terms together and all terms with ***y*** terms together (using brackets).

$$(x^2 - 4x\ ) \quad + \quad (y^2 - 6y\ ) \quad + 9 = 0$$

We now complete the square within each bracket and compensate by adding 13 to other side also:

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) + 9 = 0 + 13$$

This gives us : $(x^2 - 4x + 4) + (y^2 - 6y + 9) = 4$ [Adding – 9 to both sides]

Now we simplify and write in standard form: $(x - 2)^2 + (y - 3)^2 = 4$ **OR** $(x - 2)^2 + (y - 3)^2 = 2^2$

Comparing this equation with the standard equation:

centre of circle at $\boldsymbol{c}\ (\boldsymbol{h}, \boldsymbol{k}) = \boldsymbol{c}\ (\mathbf{2}, \mathbf{3})$ and radius of circle = $\boldsymbol{r} = 2$.

Method 2: This method involves using formulae (on previous page).
Equation of circle: $x^2 + y^2 - 4x - 6y + 9 = 0$ implies that $\boldsymbol{g} = -2$, $\boldsymbol{f} = -3$ and $\boldsymbol{c} = 9$.

Thus, the ***centre*** of this circle is: $(-\boldsymbol{g}, -\boldsymbol{f}) = (\mathbf{2}, \mathbf{3})$

and the ***radius*** of this circle is: $\sqrt{\boldsymbol{g}^2 + \boldsymbol{f}^2 - \boldsymbol{c}} = \sqrt{(-2)^2 + (-3)^2 - 9} = \sqrt{4 + 9 - 9} = \sqrt{4} = 2$.

Example 4: Is the point ***q*** (3 , 4) *inside*, *outside* or *on* the circle with equation $(\boldsymbol{x} + \mathbf{6})^2 + (\boldsymbol{y} - \mathbf{1})^2 = \mathbf{85}$

Solution 4: Find the distance from the centre of the circle to ***q*** and compare this with the length of the radius.

Method 1: centre $\boldsymbol{c} = (-\boldsymbol{g}, -\boldsymbol{f}) = (-6, 1)$; radius $\boldsymbol{r} = \sqrt{85} = 9.230$

distance from ***c*** to ***q*** = $\sqrt{(-6-3)^2 + (1-4)^2} = \sqrt{81 + 9} = \sqrt{90} = 9.487$

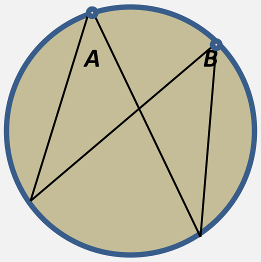
Since the distance from ***c*** to ***q*** is greater than the radius, point ***q*** is outside the circle.

Method 2: Substitute the point (3 , 4) in for $\boldsymbol{x}$ and $\boldsymbol{y}$ in the LHS of the circle equation
$(\boldsymbol{x} + \mathbf{6})^2 + (\boldsymbol{y} - \mathbf{1})^2 = \mathbf{85}$.
LHS = $(3 + 6)^2 + (4 - 1)^2 = 81 + 9 = 90$
since **LHS > RHS**, point $(3, 4)$ is **outside** the circle.

Note: Point $(3, 3)$ is **on** the circle $(\boldsymbol{x} + \mathbf{6})^2 + (\boldsymbol{y} - \mathbf{1})^2 = \mathbf{85}$, since **LHS** $= 81 + 4 = 85 =$ **RHS**

Point $(3, 2)$ is **inside** the circle $(\boldsymbol{x} + \mathbf{6})^2 + (\boldsymbol{y} - \mathbf{1})^2 = \mathbf{85}$, since **LHS** $= 81 + 1 = 82$ **< RHS**

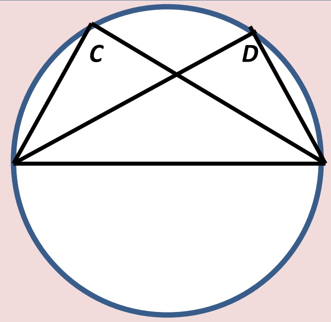
Some important Junior Cert Theorems



Angles standing on the.

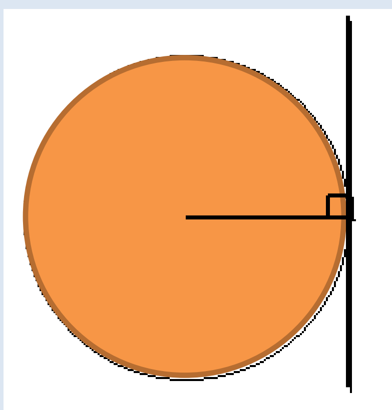
same arc of a circle are equal

$|angle\ \boldsymbol{A}| = |angle\ \boldsymbol{B}|$



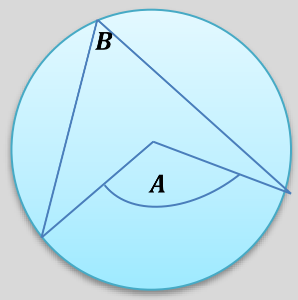
Angles formed by drawing lines from the ends of the diameter of a circle to its circumference form a right angle. So $\boldsymbol{C} = \boldsymbol{D} = \boldsymbol{90^o}$.

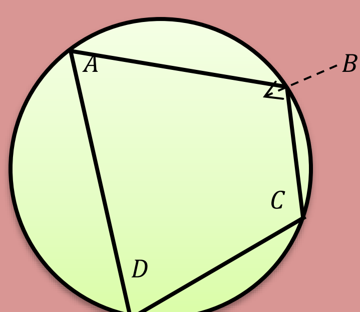
A tangent to a circle forms a right angle with the circle's radius, at the point of contact of the tangent.



The angle formed at the centre of a circle by lines originating from two points on the circle's circumference is double the angle formed on the circumference of the circle by lines originating from the same points.

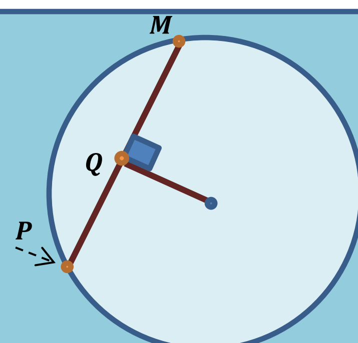
i.e. $\boldsymbol{A} = \boldsymbol{2B}$





A **cyclic quadrilateral** is a four-sided figure inside a circle, with each vertex of the quadrilateral touching the circumference of the circle. The opposite angles of such a quadrilateral add up to 180º.

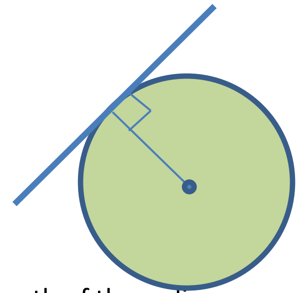
i.e. $\boldsymbol{A} + \boldsymbol{C} = 180^o$ and $\boldsymbol{B} + \boldsymbol{D} = 180^o$



A radius which is perpendicular to a chord of a circle will always **bisect** that chord.

i.e. $|PQ| = |QM|$

Tangents



Tangents are lines that touch a circle at one point only.

As can be seen in the figure across, the tangent line is always at right angles to the radius at the point of contact.

Also, the perpendicular distance from the centre of a circle to any tangent is equal to the length of the radius.

Example 5: By finding the point of intersection, show that the line $\mathbf{7x - 9y + 130 = 0}$ is a tangent to the circle $\mathbf{x^2 + y^2 = 130}$.

Solution 5: If $7x - 9y + 130 = 0$, then $x = \frac{9y-130}{7}$

$\Rightarrow \left(\frac{9y-130}{7}\right)^2 + y^2 = 130$ [Using $x^2 + y^2 = 130$]

$\Rightarrow (9y - 130)^2 + 49\,y^2 = 6370$ [Multiplying across by 49]

$\Rightarrow 81\,y^2 - 2340y + 16900 + 49y^2 = 6370$ [Expanding LHS]

$\Rightarrow 130\,y^2 - 2340y + 10530 = 0$ [Gathering like terms]

$\Rightarrow y^2 - 18y + 81 = 0$ [Dividing across by 130]

$\Rightarrow (y-9)(y-9) = 0$ [Factorising]

$\Rightarrow$ y = 9 $\Rightarrow x = \frac{9y-130}{7} = -7$ [Solving]

$\Rightarrow$ There is just a single point of intersection $(-7, 9)$

$\Rightarrow$ $\mathbf{7x - 9y + 130 = 0}$ is a tangent to the circle $\mathbf{x^2 + y^2 = 130}$.

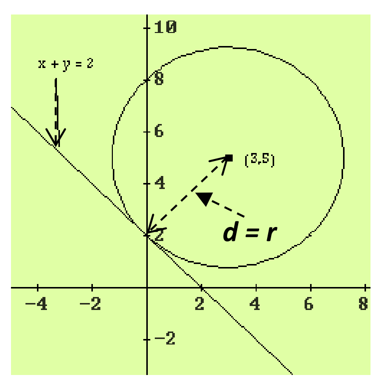
Note:
If there are **two** solutions for the intersection of a line and a circle, then the line **cuts** the circle in two places.
If there are **no** solutions for the intersection of a line and a circle, then the line does NOT meet the circle at all.

Example 6: Find the equation of the circle that has its centre at $(3, 5)$ and has a tangent whose equation is given by $x + y = 2$.

Solution 6: Perpendicular distance from $(\mathbf{3, 5})$ to $\boldsymbol{x + y = 2}$ is given by $d = \left|\frac{a\,x_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

$a = 1$	$b = 1$	$c = -2$
$x_1 = 3$	$y_1 = 5$	

<-- - - [Note minus sign here]



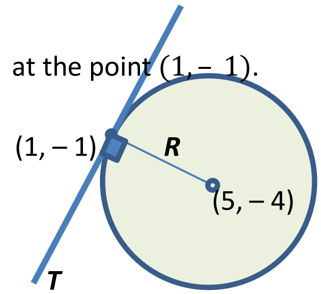
$\boldsymbol{d} = \left|\frac{(1)(3) + (1)(5) + (-2)}{\sqrt{1^2 + 1^2}}\right|$

$= \left|\frac{6}{\sqrt{2}}\right| = 3\sqrt{2}$ = **length of radius**

$\therefore$ **equation of circle:** $\mathbf{(x - 3)^2 + (y - 5)^2 = (3\sqrt{2})^2 = 18}$

Note: Solution is drawn in graphical form in the diagram above. This is NOT required by the question.

Example 7: Find the equation of the tangent to the circle $(x-5)^2+(y+4)^2=25$ at the point $(1,-1)$.



Solution7: Slope of radius, ***R***, = $\frac{y_2-y_1}{x_2-x_1} = \frac{-4+1}{5-1} = \frac{-3}{4}$

$\Rightarrow$ slope of tangent, ***T***, = $\frac{4}{3}$ [because $T \perp R$]

$\Rightarrow$ equation of tangent, ***T***, = $y-y_1 = m(x-x_1)$

$\Rightarrow$ eqn of ***T***: $y+1 = \frac{4}{3}(x-1) \Rightarrow 3(y+1) = 4(x-1)$

$\Rightarrow$ $3y+3 = 4x-4$

$\Rightarrow$ ***T***: $4x-3y-7 = 0$

Example 8: Find the equations of the two tangents from the point $(3,5)$ to the circle $x^2+y^2+2x-4y-4=0$.

Solution 8: Equation of any line containing the point $(3,5)$ is: $y-5 = m(x-3) \Rightarrow mx-y+5-3m = 0$

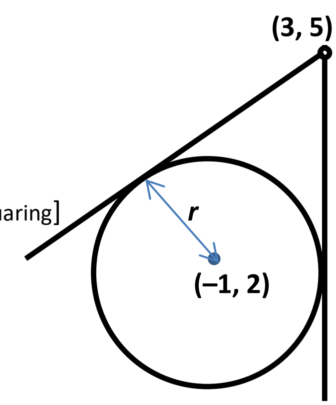
Centre of circle = $(-1,2)$ Radius of circle = $\sqrt{1^2+(-2)^2-(-4)} = \sqrt{9} = 3$

Distance from centre= $(-1,2)$ to tangent $mx-y+5-3m=0$ is equal to length of radius $=3$.

We will use the perpendicular distance from a point to a line formula: $d = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

a = m	b = –1	$c = 5-3m$
x_1 = – 1	y_1 = 2	

$d = \left|\frac{(m)(-1)+(-1)(2)+(5-3m)}{\sqrt{m^2+(-1)^2}}\right|$



$\Rightarrow d = \left|\frac{-m-2+5-3m}{\sqrt{m^2+1}}\right| \Rightarrow 3 = \left|\frac{-4m+3}{\sqrt{m^2+1}}\right|$

$\Rightarrow 9(m^2+1) = (3-4m)^2$ [Cross multiplying and squaring]

$\Rightarrow 9m^2+9 = 9-24m+16m^2$

$\Rightarrow 0 = 7m^2-24m \Rightarrow m(7m-24) = 0$

$\Rightarrow m = 0 \quad or \quad m = \frac{24}{7}$

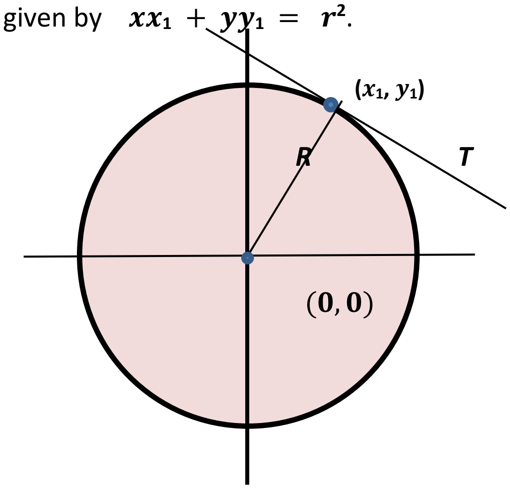
<u>If *m* = 0</u>: eqn of tangent: $-y+5=0 \quad or \quad y=5$

<u>If *m*</u> = $\frac{24}{7}$: eqn of tangent: $\frac{24}{7}x-y+5-3\left(\frac{24}{7}\right)=0 \quad or \quad 24x-7y+35-72=0$

$\therefore \; y = 5 \quad or \quad 24x-7y-37=0$

Proof of the Tangent Theorem

Prove that the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) is given by $xx_1 + yy_1 = r^2$.



Proof:

Let ***R*** be the radius of the circle, joining the centre $(0, 0)$ to (x_1, y_1).

Let ***T*** be the tangent of the circle, whose equation we are trying to find.

Slope of radius, ***R***, = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1}$

$\Rightarrow$ slope of tangent, ***T***, = $\frac{-x_1}{y_1}$

Equation of tangent, ***T***, = $y - y_1 = m(x - x_1)$

$\Rightarrow$ equation of ***T*** : $y - y_1 = \frac{-x_1}{y_1}(x - x_1)$

$\Rightarrow$ $yy_1 - y_1^2 = -xx_1 + x_1^2$

$\Rightarrow$ $xx_1 + yy_1 = x_1^2 + y_1^2$

But, (x_1, y_1) is a point on the circle $x^2 + y^2 = r^2 \Rightarrow x_1^2 + y_1^2 = r^2$

Hence, the equation of the tangent, ***T***, is given by: $xx_1 + yy_1 = r^2$.

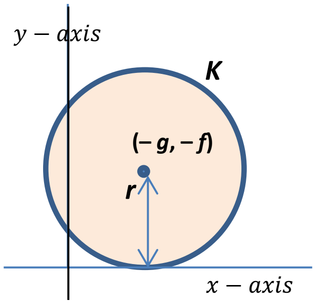
Problems involving g, f and c

When asked to find the equation of a circle which obeys certain conditions, it is usually better to write the circle in the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

Then apply the various conditions given to find the values of g, f and c.

Some special cases that apply (you should try to justify each of them yourself, one is done for you below)

- **(i)** If (0, 0) is on the circle, then $c = 0$
- **(ii)** If the centre of the circle is on the *x*-axis, then $f = 0$
- **(iii)** If the centre of the circle is on the *y*-axis, then $g = 0$
- **(iv)** If the *x*-axis is a tangent to the circle, then $r = |f| \Rightarrow c = g^2$ (see diagram below)
- **(v)** If the *y*-axis is a tangent to the circle, then $r = |g| \Rightarrow c = f^2$
- **(vi)** If both axes are tangents to the circle, then $r = |g| = |f| \Rightarrow c = g^2 = f^2$
- **(vii)** If the centre of the circle lies on the line $ax + by + c = 0$,
  then $a(-g) + b(-f) + c = 0 \Rightarrow ag + bf - c = 0$
- **(viii)** If (x_1, y_1) is a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$,
  then $(x_1)^2 + (y_1)^2 + 2g(x_1) + 2f(y_1) + c = 0$.



In this diagram, the x-axis is a tangent to the circle K, $x^2 + y^2 + 2gx + 2fy + c = 0$.

From the diagram, it is clear that the radius of the circle, $r = |f|$.

From our formula, we know that the radius of the circle, $r = \sqrt{g^2 + f^2 - c}$

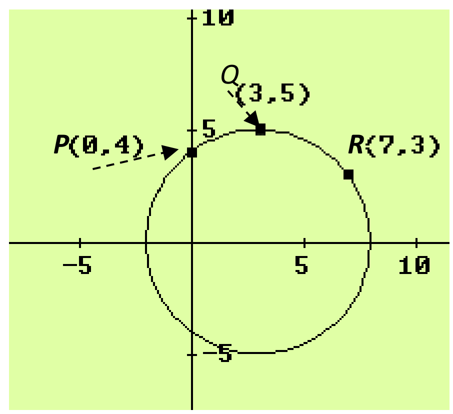
Therefore, $|f| = \sqrt{g^2 + f^2 - c}$ $\Rightarrow f^2 = g^2 + f^2 - c$

$\Rightarrow 0 = g^2 - c$

$\Rightarrow c = g^2$

Example 9: Find the equation of the circle such that the three points $P(0,4)$, $Q(3,5)$ and $R(7,3)$ are on the circle.

Solution 9: Let equation of required circle, K, be: $x^2 + y^2 + 2gx + 2fy + c = 0$



$(0, 4) \in K \Rightarrow 0^2 + 4^2 + 2g(0) + 2f(4) + c = 0$

$\Rightarrow 8f + c = -16$ ... eqn (i)

$(3, 5) \in K \Rightarrow 3^2 + 5^2 + 2g(3) + 2f(5) + c = 0$

$\Rightarrow 6g + 10f + c = -34$ ... eqn (ii)

$(7, 3) \in K \Rightarrow 7^2 + 3^2 + 2g(7) + 2f(3) + c = 0$

$\Rightarrow 14g + 6f + c = -58$ ... eqn (iii)

7 × (eqn (ii)) : $42g + 70f + 7c = -238$

3 × (eqn (iii)): $42g + 18f + 3c = -174$

Subtracting: $52f + 4c = -64$ ... eqn (iv)

4 × (eqn (i)) : $32f + 4c = -64$

$\Rightarrow 20f = 0$

$\Rightarrow f = 0 \Rightarrow c = -16$

If $f = 0$, and $c = -16$,

then eqn (ii): $6g = -34 - 10f - c$

$6g = -34 - (10)(0) - (-16) = -18$

$\Rightarrow g = -3$

∴ Required equation: $x^2 + y^2 - 6x - 16 = 0$ **or** $(x - 3)^2 + y^2 = 25$

Example 10: Find the equation of the circle such that the two points (1 , 0), and (0 , 2) are on the circle and the centre of the circle lies on the line, ***D:*** $x + 3y - 11 = 0$.

Solution 10: Let equation of required circle, ***K***, be: $x^2 + y^2 + 2gx + 2fy + c = 0$

(1, 0) ε ***K*** ⇒ $1^2 + 0^2 + 2g(1) + 2f(0) + c = 0$ [Substituting given point into circle equation]

⇒ $2g + c = -1$ … eqn (i)

(0, 2) ε ***K*** ⇒ $0^2 + 2^2 + 2g(0) + 2f(2) + c = 0$ [Substituting given point into circle equation]

⇒ $4f + c = -4$ … eqn (ii)

($-g$, $-f$) ε ***D*** ⇒ $-g - 3f - 11 = 0$ [Centre of circle is ($-g$, $-f$) lies on line $x + 3y - 11 = 0$]

⇒ $g + 3f = -11$ … eqn (iii)

(eqn (i)) : $2g + c = -1$

(eqn (ii)): $4f + c = -4$

Subtracting: $2g - 4f = 3$ … eqn (iv)

But, 2 × (eqn (iii)): $2g + 6f = -22$

⇒ $-10f = 25$

⇒ $f = -2.5$

Using eqn (iii): $g = -11 - (3)(-2.5)$

⇒ $g = -3.5$

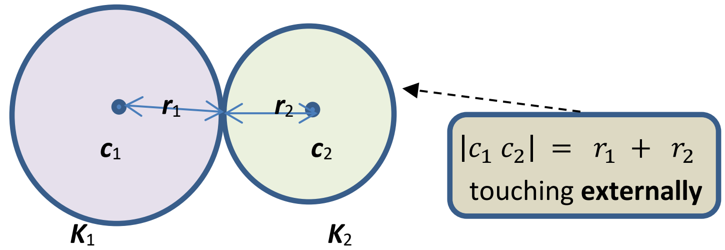
If $f = -2.5$ and $g = -3.5$,

then eqn (i): $c = -1 - (2)(-3.5) = 6$

∴ Required equation: $x^2 + y^2 - 7x - 5y + 6 = 0$ **or** $(x - \frac{7}{2})^2 + (y - \frac{5}{2})^2 = \frac{25}{2}$.

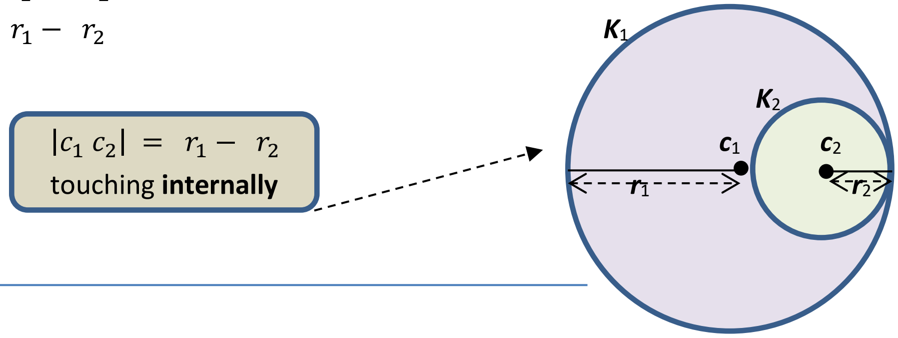
Touching Circles, Common Tangent

If c_1 and c_2 are the centres of two circles, K_1 and K_2, and r_1 and r_2 are the radii of these two circles then, these circles:



touch **externally** iff $\quad |c_1\,c_2| = r_1 + r_2$

and touch **internally** iff $\quad |c_1\,c_2| = r_1 - r_2$



Example 11: Prove that the two circles K_1: $x^2 + y^2 - 6x - 8y + 24 = 0$ and K_2: $x^2 + y^2 = 16$ touch externally and find their point of contact.

Solution 11: $c_1 = (3, 4) \quad ; \quad c_2 = (0, 0)$

$r_1 = \sqrt{3^2 + 4^2 - 24} = 1$ and $r_2 = 4$.

$|c_1\,c_2| = \sqrt{3^2 + 4^2} = 5$

$r_1 + r_2 = 1 + 4 = 5$

Therefore, as $|c_1\,c_2| = r_1 + r_2$, the circles touch externally.

Note: $K_1 - K_2 = (x^2 + y^2 - 6x - 8y + 24) - (x^2 + y^2 - 16) = 0$

$\Rightarrow -6x - 8y + 40 = 0 \quad \Rightarrow \quad 3x + 4y - 20 = 0$

[This is the equation of the **common tangent**. It is got by subtracting the two circle equations. This method will give you the **common chord** when two circles cut each other in **two** places.]

Now find where $3x + 4y - 20 = 0$ meets, say, K_2, $x^2 + y^2 = 16$

Using simultaneous equations, we find the point of intersection by substituting

$x = \frac{20-4y}{3}$ into $x^2 + y^2 = 16$ and finding a single solution, i.e. $y = \frac{16}{5}$, $x = \frac{12}{5}$

Therefore, point of intersection or contact: $\left(\frac{12}{5} , \frac{16}{5} \right)$.

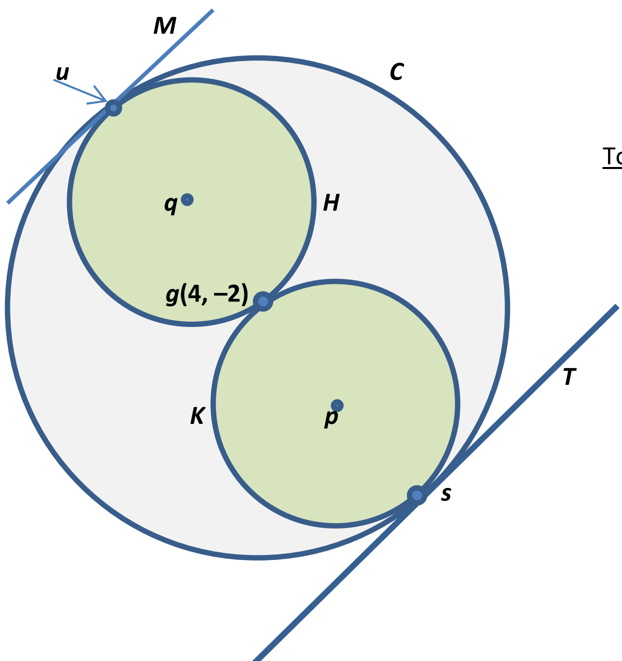
Example 12: (i) ***C*** is the circle $x^2 + y^2 - 8x + 4y - 5 = 0$. A circle ***K*** touches ***C*** internally and goes through ***g***, the centre of ***C***. If $3x - 4y + 5 = 0$ is the tangent common to both circles, find the equation of ***K***.

(ii) Circle ***H*** is the image of circle ***K*** under the central symmetry in ***g***.
Find the equation of ***H*** and the equation of the common tangent to ***H*** and ***C***.

Solution 12: (i) Equation of ***C***: $x^2 + y^2 - 8x + 4y - 5 = 0$ $\Rightarrow$ centre of $C = g = (4,\ 2)$

$\Rightarrow$ radius of $C = \sqrt{-4^2 + 2^2 - (-5)} = \sqrt{25} = 5$

$\Rightarrow$ radius of $K = \frac{1}{2}(5) = \frac{5}{2}$



<u>To find ***s***</u>: $T \cap C = s$

T: $3x - 4y + 5 = 0 \Rightarrow x = \frac{4y-5}{3}$

C: $x^2 + y^2 - 8x + 4y - 5 = 0$

$(\frac{4y-5}{3})^2 + y^2 - 8(\frac{4y-5}{3}) + 4y - 5 = 0$

$\Rightarrow (4y-5)^2 + 9y^2 - 24(4y-5) + 36y - 45 = 0$

$\Rightarrow 16y^2 - 40y + 25 + 9y^2 - 96y + 120 + 36y - 45 = 0$

$\Rightarrow 25y^2 - 100y + 100 = 0$

$\Rightarrow y^2 - 4y + 4 = 0$

$\Rightarrow (y-2)^2 = 0 \Rightarrow y = 2$

$\Rightarrow x = \frac{4y-5}{3} = \frac{4(2)-5}{3} = 1$

$\Rightarrow$ point of intersection, ***s*** = **(1, 2)**

<u>To find equation of circle ***K***</u>

Centre of ***K*** = ***p*** = mid-point of $[rs] = \left(\frac{4+1}{2}, \frac{-2+2}{2}\right) = \left(\frac{5}{2}, 0\right)$; radius of $K = \frac{5}{2}$

<u>Equation of ***K***</u>: $(x - \frac{5}{2})^2 + (y-0)^2 = (\frac{5}{2})^2 \Rightarrow x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$

$\Rightarrow x^2 + y^2 - 5x = 0$

(ii) <u>To find equation of circle ***H***</u>

H is the image of ***K*** under the central symmetry in ***g***. $\therefore$ radius of ***H*** = radius of $K = \frac{5}{2}$

$\therefore$ ***q*** is the image of ***p*** under the central symmetry in ***g***. $p\left(\frac{5}{2}, 0\right) \rightarrow g\,(4, -2) \rightarrow q\,(\frac{11}{2}, -4)$

<u>Equation of ***H***</u>: $(x - \frac{11}{2})^2 + (y+4)^2 = (\frac{5}{2})^2 \Rightarrow x^2 - 11x + \frac{121}{4} + y^2 + 8y + 16 = \frac{25}{4}$

$\Rightarrow x^2 + y^2 - 11x + 8y + 40 = 0$

<u>To find equation of line ***M*** (see diagram)</u>

Line ***M*** is parallel to line ***T*** and contains the point ***u*** (see diagram). $\therefore$ Slope of ***M*** = slope of $T = \frac{3}{4}$.

u is the image of ***g*** under the central symmetry in ***q***. $g\,(4, -2) \rightarrow q\,(\frac{11}{2}, -4) \rightarrow u\,(7, -6)$

$\therefore$ equation of ***M*** : $y - y_1 = m(x - x_1)$

$\Rightarrow y + 6 = \frac{3}{4}(x - 7)$

$\Rightarrow 4(y+6) = 3(x-7)$

$\Rightarrow 4y + 24 = 3x - 21$

$\Rightarrow 3x - 4y - 45 = 0$

Note: The common tangent, ***M*** can also be found using: $M = C - H$ [Intersection of circles ***C*** and ***H***]

$\therefore$ **equation of** $M = (x^2 + y^2 - 8x + 4y - 5) - (x^2 + y^2 - 11x + 8y + 40) = 0$

$\therefore$ **equation of** $M = 3x - 4y - 45 = 0$

POINTS TO REMEMBER

1) Check the **formulae** available on *page* ***19*** and *page* ***9*** of your **Mathematical Tables** and Formulae booklet.

2) Drawing a quick sketch/diagram can often be a great help with problems on the circle.
For accurate circles you have to make the gaps on each axis the same.

3) Realise that you may need a lot of your formulae from "the straight line" here (*page* ***18*** Tables).

4) **Length of arc** of circle, $\boldsymbol{l = r\,\theta}$ *(*)* and **area of sector** $\boldsymbol{A = \frac{1}{2}\, r^2\, \theta}$ (*) [angle, θ, is given in radians]
(Also, $l = \frac{\theta}{360}(2\pi r)$ is used when the angle, θ, is given in degrees.)
(Also, the equation $A = \frac{\theta}{360}\,(\pi r^2)$ is used when the angle, θ, is given in degrees.)

5) ***C:*** $\boldsymbol{x^2 + y^2 = r^2}$ (**) represents a circle of centre $(0, 0)$ and radius $\boldsymbol{r}$.

6) $\boldsymbol{(x - h)^2 + (y - k)^2 = r^2}$ *(*)* represents a circle of centre $(\boldsymbol{h}, \boldsymbol{k})$ and radius $\boldsymbol{r}$.

7) The equation of a circle should be of **degree** $\mathbf{2}$, can**not** have an $\boldsymbol{xy}$ term and **coefficients** of x^2 and y^2 must be **equal**.

8) When the equation of a circle is given in the form: $\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$ *(*)*
the centre of the circle is: $(-\boldsymbol{g}, -\boldsymbol{f})$ (*) and the radius is given by: $\boldsymbol{r = \sqrt{g^2 + f^2 - c}}$ **(*).**

9) A **point** is **inside** a circle if its distance from the centre of the circle is **less than the radius** of the circle.

10) A **point** is **on** a circle if its distance from the centre of the circle is **equal to the radius** of the circle.

11) A **point** is **outside** a circle if its distance from the centre of the circle **is greater than the radius** of the circle.

12) **Angles** standing on the **same arc** of a circle are **equal**.

13) The **angle at a circle** standing on a **diameter** is **a right angle**.

14) A **tangent** to a circle **is perpendicular to the radius** drawn to the point of contact.

15) The **angle at the centre of a circle is twice the angle at the circle**, which is standing on the same arc.

16) In a **cyclic quadrilateral**, opposite angles add up to **180º**.

17) A **radius which is perpendicular to a chord** of a circle **bisects** that chord.

18) A line drawn **perpendicularly** from the **centre** of a circle on to any **chord** will **bisect** that chord.

19) The tangent to the circle $\boldsymbol{x^2 + y^2 = r^2}$ at the point $\boldsymbol{(x_1, y_1)}$ is given by $\boldsymbol{xx_1 + yy_1 = r^2}$ (not needed).

20) The **perpendicular distance** from **the centre** of a circle to **any tangent** of that circle is equal to **the radius**.

21) The **slope of a tangent** will be the **negative inverse** of the **slope of the radius** to the circle at the point of contact.

22) When finding the equation of two tangents from a given point outside a given circle, let ***m*** be the required slope of the tangent, find the equation of the tangent in terms of ***m***, and use the fact that the perpendicular distance from the centre of the circle to the tangent is equal to the radius.

23) When solving equations using the equation $\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$, it is useful to know that

(i) if (0, 0) is on the circle, then $\boldsymbol{c} = \mathbf{0}$

(ii) **i**f the centre of the circle is on the *x*-axis, then $\boldsymbol{f} = \mathbf{0}$; if the centre of the circle is on the *y*-axis, then $\boldsymbol{g} = \mathbf{0}$

(iii)if the *x*-axis is a tangent to the circle, then $\boldsymbol{r} = |\boldsymbol{f}| \Rightarrow \boldsymbol{c} = \boldsymbol{g}^2$

(iv)if the *y*-axis is a tangent to the circle, then $\boldsymbol{r} = |\boldsymbol{g}| \Rightarrow \boldsymbol{c} = \boldsymbol{f}^2$

(v) if both axes are tangents to the circle, then $\boldsymbol{r} = |\boldsymbol{g}| = |\boldsymbol{f}| \Rightarrow \boldsymbol{c} = \boldsymbol{g}^2 = \boldsymbol{f}^2$

(vi)if the centre of the circle lies on the line $\boldsymbol{ax + by + c = 0}$, then $a(-g) + b(-f) + c = 0 \Rightarrow ag + bf - c = 0$.

24) When circles **touch externally**, $|\boldsymbol{c_1\, c_2}| = \boldsymbol{r_1 + r_2}$. (**)

25) When circles **touch internally**, $|\boldsymbol{c_1\, c_2}| = \boldsymbol{r_1 - r_2}$, (**) where $\boldsymbol{r_1}$ is the radius of the bigger circle.

NOTES