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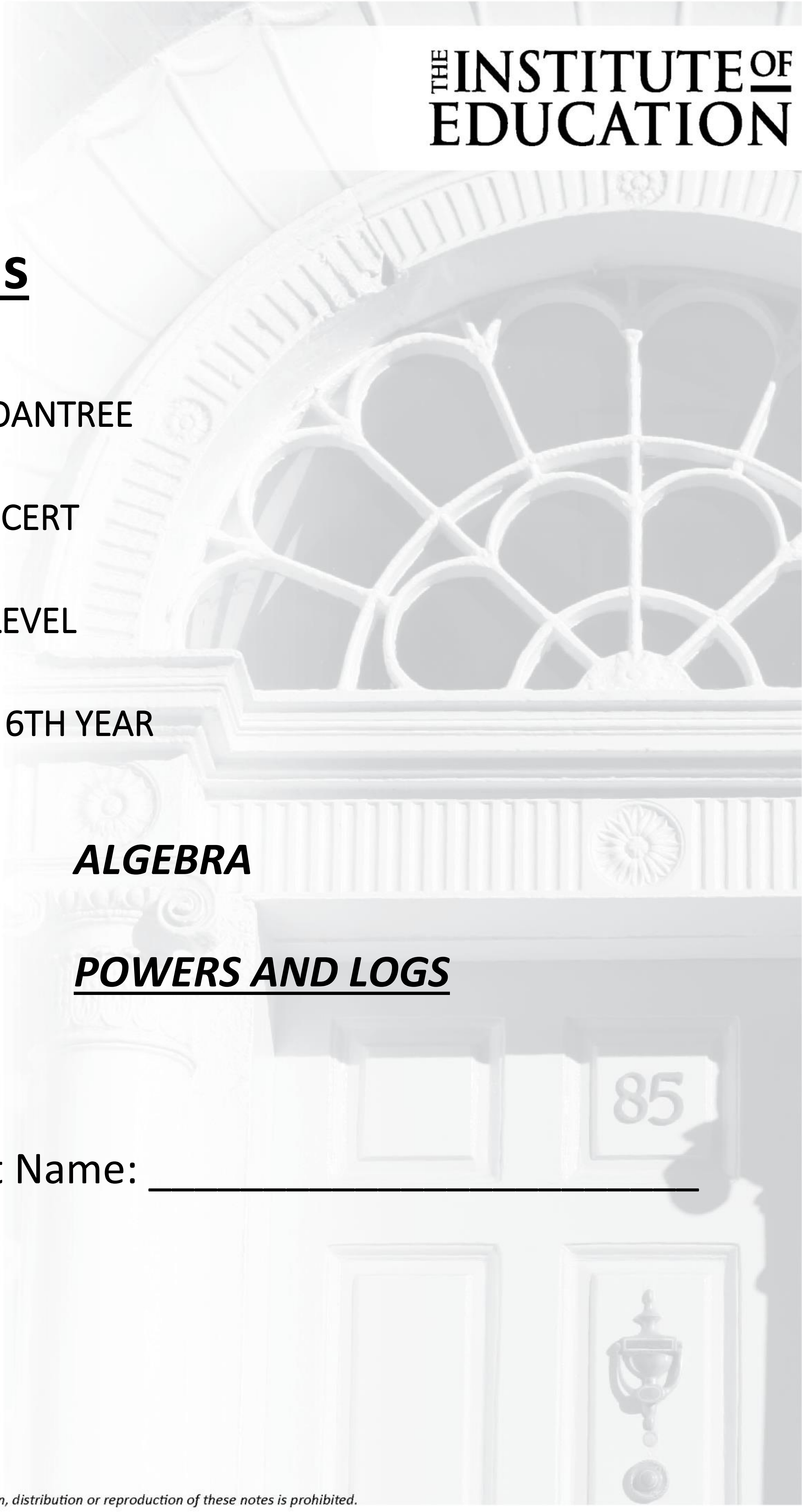

Maths

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LEAVING CERT

HIGHER LEVEL

5TH AND 6TH YEAR

TOPIC: ALGEBRA

UNIT 8: POWERS AND LOGS

Student Name: ______________________________

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CONTENTS

UNIT 8: POWERS AND LOGS

Unit 8 Algebra

Powers and Logs

8.1 Powers and the Laws of Indices

1. **Definition of Powers and Indices**
In the statement

$$2^3 = 8$$

* 2^3 is called a **power**, i.e. '2 to the power of 3',
* 2 is the **base** of the power,
* 3 is the **index**, or **exponent**, of the power (although it is often seen that 3 is referred to as a 'power', which can cause confusion),
* 8 is the **value** of the power.

2. **Laws of Indices**
The rules for dealing with powers are called the Laws of Indices. They are given on page 21 of the *Formulae and Tables*.

Laws of Indices

1.	$a^p.a^q = a^{p+q}$,	e.g. $3^5.3^2 = 3^{5+2} = 3^7$
2.	$\dfrac{a^p}{a^q} = a^{p-q}$	e.g. $\dfrac{4^8}{4^3} = 4^{8-3} = 4^5$
3.	$\left(a^p\right)^q = a^{pq}$	e.g. $\left(x^2\right)^{n+1} = x^{2(n+1)} = x^{2n+2}$
4.	$a^0 = 1$	e.g. $3^0 = 1$
5.	$a^{-p} = \dfrac{1}{a^p}$	e.g. $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$
6.	$a^{\frac{1}{q}} = \sqrt[q]{a}$	e.g. $a^{\frac{1}{3}} = \sqrt[3]{a}$
7.	$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$	e.g. $27^{\frac{4}{3}} = \sqrt[3]{27^4} = \left(\sqrt[3]{27}\right)^4$
8.	$(ab)^p = a^p b^p$	e.g. $(3x)^4 = 3^4 x^4 = 81x^4$
9.	$\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$	e.g. $\left(\dfrac{2x}{y}\right)^4 = \dfrac{(2x)^4}{y^4} = \dfrac{16x^4}{y^4}$

Example 1

Express each of the following in the form 2^k, where $k \in \mathbb{R}$.

(i) 64 (ii) $\dfrac{\sqrt{32}}{4}$ (iii) $4^x.8^{2x-1}$ (iv) $\sqrt{128^{1-2x}}$.

Solution

(i) $64 = 2\times2\times2\times2\times2\times2 = 2^6$

(ii) $$\frac{\sqrt{32}}{4} = \frac{(2^5)^{\frac{1}{2}}}{2^2} = \frac{2^{\frac{5}{2}}}{2^2} = 2^{\frac{5}{2}-2} = 2^{\frac{1}{2}}$$

(iii) $$4^x.8^{2x-1} = (2^2)^x(2^3)^{2x-1} = 2^{2x}.2^{6x-3} = 2^{8x-3}$$

(iv) $$\sqrt{128^{1-2x}} = \left((2^7)^{1-2x}\right)^{\frac{1}{2}} = \left(2^{7-14x}\right)^{\frac{1}{2}} = 2^{\frac{7}{2}-7x}.$$

3. The Number ‘*e*’

* The number *e* is a famous and important mathematical constant whose value is approximately
  $e = 2\cdot718$.
* The origin of this number will be seen in a later chapter.
* Powers with a base of *e* will occur frequently on our course.
* It is important to remember that powers with a base of *e* obey all the usual laws of indices.

Example 2

Express as a single power:

(i) $\dfrac{e^{2x-1}.e^{3x+2}}{e^{4-5x}}$ (ii) $\sqrt{\dfrac{1}{e^{3x+7}}}$.

Solution

(i) $\dfrac{e^{2x-1}.e^{3x+2}}{e^{4-5x}} = \dfrac{e^{5x+1}}{e^{4-5x}}$

$= e^{(5x+1)-(4-5x)} = e^{10x-3}$

(ii) $\sqrt{\dfrac{1}{e^{3x+7}}} = \left(e^{-(3x+7)}\right)^{\frac{1}{2}}$

$= e^{-\frac{1}{2}(3x+7)}$

$= e^{-\frac{3}{2}x-\frac{7}{2}}$.

Exercises 8.1

Simplify each of the following, without using a calculator:

1. $2^5 \times 2^{-3}$

2. $\dfrac{3^5}{(3^{-2})^2}$

3. $\dfrac{\sqrt{8}\sqrt{32}}{2^{-3}}$

4. $a^{2x+1}.a^{3x+1}$

5. $\dfrac{(a^{x+1})^4}{a^{2-x}}$

6. $\sqrt{\dfrac{b^{3x-1}}{b^{x+5}}}$

7. $\left(a^2b^3\right)^{-3}$

8. $\dfrac{xy^3}{(x^2y)^{-1}}$

9. $\left(\dfrac{p^2q}{q^4}\right)^2$

10. Evaluate $5(4^{3n+1}) - 20(8^{2n})$.

11. If $f(n) = 4(2^n)$, show that $f(n+k) = 2^k f(n)$.

12. Express $2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}}$ in the form 2^k, without using a calculator.

13. Express in the form e^{ax+b}:

(i) $e^{3x+2}.e^{4-2x}$ (ii) $\left(e^{2x-1}\right)^2.e^{5x+7}$

(iii) $\dfrac{e^{2x+5}}{e^{1-3x}}$ (iv) $\dfrac{\left(e^{1-3x}\right)^3}{e^{2-5x}}$

14. Express each of the following as a single power with a base of e:

(i) $\dfrac{e^{x^2-2x+4}}{e^{1-3x}}$ (ii) $\sqrt{\dfrac{e^{5x-1}}{e^{3-7x}}}$

(iii) $\dfrac{\sqrt{e^{3x+1}}}{\left(e^{2-x}\right)^2}$ (iv) $\dfrac{e^{x^2-x+4}}{\sqrt{e^{2x+6}}}$

15. Show that $\left(e^x+\dfrac{1}{e^x}\right)^2=e^{2x}+e^{-2x}+2$.

16. Simplify $\dfrac{e^{2x}+e^{x+2}}{e^{x+1}}$.

8.2 Logarithms

1. Definition of Logarithm (Log)

* Logarithm, or log, is another term for index, when the emphasis is placed on the index.
* For example, in the index statement

  $2^3 = 8$,

  the log is 3.

2. Index Statement and Log Statement

* The index statement

  $2^3 = 8$

  can be written as a log statement in the form

  $\log_2 8 = 3$,

  i.e. the base, 2, is written as a subscript and the value of the power, 8, is written directly after the log, and the value of the log (index) is 3.

* The meaning of the symbol

  $\log_a x$

  is the index that must be placed above the base *a* to give the value *x*.

Definition of Logs

$$a^y = x \qquad \Leftrightarrow \qquad \log_a x = y$$

* It is important to be able to convert between an index statement ($a^y = x$) and the corresponding log statement ($\log_a x = y$).

3. Laws of Logs

Laws of Logs

1. $\log_a (xy) = \log_a x + \log_a y$

2. $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a x^q = q \log_a x$

4. $\log_a a = 1$ and $\log_a 1 = 0$

5. $\log_a x = \dfrac{\log_b x}{\log_b a}$ **(Change of Base Law)**

The Laws of Logs are related to the Laws of Indices and can be derived from these laws. These laws are often used in conjunction with each other.

Example 1

If $x = \log_{10} 2$ and $y = \log_{10} 3$, express in terms of x and y:

(i) $\log_{10} \sqrt{\dfrac{32}{27}}$, (ii) $\log_{10} \dfrac{4000}{9}$.

Solution

(i) $\log_{10} \sqrt{\dfrac{32}{27}} = \log_{10} \left(\dfrac{32}{27}\right)^{\frac{1}{2}}$

$= \dfrac{1}{2} \log_{10} \dfrac{32}{27}$ ... *Law 3*

$= \dfrac{1}{2} \left(\log_{10} 32 - \log_{10} 27\right)$ ... *Law 2*

$= \dfrac{1}{2} \left(\log_{10} 2^5 - \log_{10} 3^3\right)$

$= \dfrac{1}{2} \left(5\log_{10} 2 - 3\log_{10} 3\right)$ ... *Law 3*

$= \dfrac{1}{2} \left(5x - 3y\right)$

(ii) $\log_{10} \dfrac{4000}{9} = \log_{10} 4000 - \log_{10} 9$ ... *Law 2*

$= \log_{10}(1000 \times 4) - \log_{10} 3^2$

$= \log_{10} 10^3 + \log_{10} 2^2 - \log_{10} 3^2$ ... *Law 1*

$= 3\log_{10} 10 + 2\log_{10} 2 - 2\log_{10} 3$ ... *Law 3*

$= 3 + 2x - 2y.$ ... *Law 4*

Example 2

(i) Show that $\log_a b = \dfrac{1}{\log_b a}$.

(ii) Show that $\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x} + \dfrac{1}{\log_c x} = \log_x (abc)$.

Solution

(i) By Law 5, the Change of Base Law,

$$\log_a b = \frac{\log_b b}{\log_b a}$$

$$= \frac{1}{\log_b a} \qquad \textit{... Law 4}$$

(ii) By part (i),

$$\frac{1}{\log_a x} = \log_x a\ , \quad \frac{1}{\log_b x} = \log_x b, \quad \frac{1}{\log_c x} = \log_x c$$

Then

$$\frac{1}{\log_a x} + \frac{1}{\log_b x} + \frac{1}{\log_c x} = \log_x a + \log_x b + \log_x c$$

$$= \log_x (a \times b \times c)$$

$$= \log_x (abc).$$

4. Natural Logs

* Logs with a base of e, which is approximately $2 \cdot 718$, are called natural logs, as they occur frequently in nature.

* An example of a natural log is

$$\log_e x\ .$$

* $\log_e$ can also be written as ‘ln’. Thus we can write

$$\log_e x = \ln x\ .$$

* Note that

$$\ln e = \log_e e = 1\ .$$

* Natural logs obey all the usual Laws of Logs, e.g.

(i) $\ln(x.y) = \ln x + \ln y$ (ii) $\ln x^4 = 4\ln x$.

Example 3

Show that

$$\ln\left(xe^{2x}\right) = 2x + \ln x\ .$$

Solution

$$\ln\left(xe^{2x}\right) = \ln x + \ln e^{2x}$$

$$= \ln x + 2x\ln e$$

$$= 2x + \ln x\ .$$

5. Using a Calculator to Evaluate Logs

* On scientific calculators, there are three separate log function buttons.
* The $\boxed{\log}$ function is used to calculate logs with a base of 10, e.g. to find $\log_{10} 7\cdot52$, press $\boxed{\log}$ $7\cdot52$ $\boxed{)}$ $\boxed{=}$ to get $0\cdot8762$.
* The $\boxed{\ln}$ function is used to calculate natural logs, i.e. $\log_e$, for example, to find $\ln 3\cdot845$, press $\boxed{\ln}$ $3\cdot845$ $\boxed{)}$ $\boxed{=}$ to get $1\cdot34677$.
* The $\boxed{\log_{\square}\square}$ function is used to calculate any log, e.g. to find $\log_{1\cdot05} 1\cdot89$, press $\boxed{\log_{\square}\square}$ $1\cdot05$ $\boxed{\rightarrow}$ $1\cdot89$ $\boxed{=}$ to get $13\cdot047$.

Exercises 8.2

1. Write each of the following as a log statement:

(i) $3^5 = 243$ (ii) $7^4 = 2401$

(iii) $1\cdot2^6 = 2\cdot985984$ (iv) $3\cdot5^{4\cdot2} = 192\cdot7905695$

2. Write each of the following as an index statement:

(i) $\log_5 125 = 3$ (ii) $\log_3 6561 = 8$

(iii) $\log_{1\cdot05} 1\cdot795856326 = 12$ (iv) $\log_{2\cdot1} 438\cdot7308937 = 8\cdot2$

Write each of the following as a single log.

3. $\log_2 x + \log_2 y$ **4.** $\log_3 a^2 - \log_3 b$

5. $\log_3 a - 2\log_3 b + 3\log_3 c$ **6.** $\log_4 x + \frac{1}{2}\log_4 y - 3\log_4 z$

7. If $x = \log_2 a$, express in terms of x:

(i) $\log_2 a^3$ (ii) $\log_2 16a^2$ (iii) $\log_2 \frac{a^4}{32}$.

8. If $p = \log_{27} q$, express in terms of p:

(i) q (ii) $\log_9 q$ (iii) $\log_q 9$.

9. If $\log_a 2 = s$ and $\log_a 3 = t$, where $a > 0$, express each of the following in terms of s and t:

(i) $\log_a 108$ (ii) $\log_a 48a^3$ (iii) $\log_a \frac{27}{32a^4}$.

10. (i) If $\log_a x = \log_a y$, show that $x = y$.

(ii) Hence show that if

$$\log_4 x = \log_2 y$$

then $x = y^2$.

11. Write each of the following as a log statement:

(i) $e^{2t} = p$ (ii) $e^{1+3x} = 4y$ (iii) $e^{x^2-x} = 3\cdot 2$.

12. Write each of the following as an index statement:

(i) $\ln(x^2 - 3x) = y$ (ii) $\ln(3x+7) = 2\cdot 8$

(iii) $\ln(5t-1) = 0\cdot 15s$ (iv) $\ln(2a+b) = 2\cdot 19$.

Simplify each of the following expressions.

13. $\log_e(x^2 e^{-x})$ **14.** $\log_e e^{2+\sin x}$ **15.** $\log_e(x^2 e^{\cos x})$

16. $\ln(x^2\sqrt{x+1})$ **17.** $\ln(x^3 e^{\sin x})$ **18.** $\ln\left(\dfrac{\sqrt{x}}{e^{4x}}\right)$

8.3 Equations Involving Laws of Indices and Logs

A Type 1: Constant Index

1. Method

To solve for a an equation such as

$$a^p = b\ ,$$

especially where p is not just 2 or 3, we can proceed as follows:

* Put both sides of the equation to the power of $\dfrac{1}{p}$:

$$a^p = b$$
$$\left(a^p\right)^{\frac{1}{p}} = (b)^{\frac{1}{p}}$$

* As the index on the LHS is now 1, we get:

$$a = b^{\frac{1}{p}}\ .$$

Example 1

Solve the equation

$$(3+2x)^{7\cdot 3} = 141\cdot 812\ ,$$

giving your answer correct to three decimal places.

Solution

$$(3+2x)^{7\cdot 3} = 141\cdot 812$$
$$3+2x = 141\cdot 812^{\frac{1}{7\cdot 3}}$$
$$3+2x = 1\cdot 9713 \qquad \text{... by calculator}$$
$$2x = -1\cdot 0287$$
$$x = -0\cdot 514$$

B Type 2: Unknown in Index: Common Base Possible

1. Common Base Possible: Single Power = Single Power

* Some equations with the unknown in the index can be written with a single power on each side of the equation.
* With some of these, it is possible to write each side with the same base.

* For example, the equation

$$2^x = \frac{1}{64}$$

can be written

$$2^x = 2^{-6}.$$

* In this case, we can put the indices equal to get

$$x = -6.$$

* Alternatively, the equation $2^x = \frac{1}{64}$ can be converted to log form:

$$x = \log_2 \frac{1}{64} = -6.$$

Example 2

Solve for $x \in \mathbb{R}$, $\quad 8^{1-2x} = \sqrt{\frac{1}{32^x}}$.

Solution

$(2^3)^{1-2x} = (32^{-x})^{\frac{1}{2}}$

$2^{3-6x} = \left((2^5)^{-x}\right)^{\frac{1}{2}}$ ... *write each side with a base of 2*

$2^{3-6x} = (2^{-5x})^{\frac{1}{2}}$

$2^{3-6x} = 2^{-\frac{5x}{2}}$ ... *now each side is a power with the same base*

$3 - 6x = -\frac{5x}{2}$ ... *equating indices*

$6 - 12x = -5x$

$6 = 7x$

$x = \frac{6}{7}$

2. Evaluating Logs Without Using a Calculator

If the base of a log and the value of the power can be written in terms of the same base, the log can be calculated without using a calculator.

Example 3

Evaluate, without using a calculator, $\log_3 \sqrt{27}$.

Solution

Let $\log_3 \sqrt{27} = x$.

Then $3^x = \sqrt{27}$

$$3^x = \left(3^3\right)^{\frac{1}{2}}$$

$$3^x = 3^{\frac{3}{2}}$$

$$x = \frac{3}{2}.$$

C Type 3: Unknown in Index: Common Base Not Possible

1. Using Logs to Solve Index Equations

* Logs can be used to solve equations with the unknown in the index, if the base and the value cannot be easily expressed as powers with the same base.

* For example, the equation

  $$7^{4x-3} = 5$$

  can be solved by using logs, because it is not possible to find a common base for the 7 and the 5.

* Two different approaches exist: converting to a log statement and taking logs on both sides. They are shown in the example below.

Example 4

Solve the equation

$$7^{4x-3} = 5$$

for $x \in \mathbb{R}$. Give your answer correct to three decimal places.

Solution

Method 1: Convert to a Log Statement

$7^{4x-3} = 5$

$\log_7 5 = 4x - 3$ ... *converting to a log statement*

$0 \cdot 8271 = 4x - 3$ ... *using the* $\log_{\square}\square$ *function on your calculator*

$4x = 3 \cdot 8271$

$x = 0 \cdot 957$

Method 2: Taking Logs (natural logs) on Both Sides

$7^{4x-3} = 5$

$\ln 7^{4x-3} = \ln 5$ ... *taking* ln *on each side*

$(4x-3)\ln 7 = \ln 5$

$4x - 3 = \dfrac{\ln 5}{\ln 7}$

$4x - 3 = 0 \cdot 8271$

$4x = 3 \cdot 8271$

$x = 0 \cdot 957$.

2. More Advanced Equations

* If an equation contains unknowns in two separate powers, with no common base possible, it is more convenient to take logs on both sides to find the solution.

* Example 5 shows one of these equations.

Example 5

Solve the equation

$$4500(1 \cdot 045)^t = 6300(1 \cdot 035)^{t+2}$$

for $t \in \mathbb{R}$. Give your answer correct to three decimal places.

Solution

$4500(1 \cdot 045)^t = 6300(1 \cdot 035)^{t+2}$

$(1 \cdot 045)^t = 1 \cdot 4(1 \cdot 035)^{t+2}$

$\ln(1 \cdot 045)^t = \ln 1 \cdot 4(1 \cdot 035)^{t+2}$

$t \ln 1 \cdot 045 = \ln 1 \cdot 4 + \ln(1 \cdot 035)^{t+2}$

$t \ln 1 \cdot 045 = \ln 1 \cdot 4 + (t+2)\ln 1 \cdot 035$

$0 \cdot 04402t = 0 \cdot 33647 + (t+2)(0 \cdot 03440)$

$0 \cdot 04402t = 0 \cdot 33647 + 0 \cdot 03440t + 0 \cdot 06880$

$0 \cdot 00962t = 0 \cdot 40527$

$t = 42 \cdot 128$

D Type 4: Equations which Become Quadratic

1. Equations with Three or More Terms

* An equation such as
$$2^{2x+3} - 33(2^x) + 4 = 0$$
can be converted to a quadratic equation by letting *y* be the basic power present, i.e.
$$y = 2^x .$$
* To write 2^{2x+3} in terms of $y = 2^x$:
$$\begin{aligned} 2^{2x+3} &= 2^{2x}.2^3 \\ &= (2^x)^2(8) \\ &= 8y^2 \end{aligned}$$
* We can now write the given equation as
$$8y^2 - 33y + 4 = 0 .$$
* We now solve this equation for *y*. Don't forget to finish by calculating the corresponding values of *x*.

Example 6

Solve the equation
$$e^{2x+1} - 14\cdot 1e^x + 2\cdot 7 = 0$$
for $x \in \mathbb{R}$. Give your answers correct to two decimal places.

Solution

Let $y = e^x$.

Then
$$\begin{aligned} e^{2x+1} &= e^{2x}.e^1 \\ &= e(e^x)^2 \\ &= ey^2 . \end{aligned}$$

Then the equation can be written:
$$ey^2 - 14\cdot 1y + 2\cdot 7 = 0 \qquad [a = e,\ b = -14\cdot 1,\ c = 2\cdot 7]$$
$$y = \frac{14\cdot 1 \pm \sqrt{(-14\cdot 1)^2 - 4(e)(2\cdot 7)}}{2(e)}$$
$$y = \frac{14\cdot 1 \pm 13\cdot 017}{2e}$$
$$y = 4\cdot 988 \quad \text{or} \quad y = 0\cdot 199$$
$$e^x = 4\cdot 988 \quad \text{or} \quad e^x = 0\cdot 199$$
$$x = \ln 4\cdot 988 \quad \text{or} \quad x = \ln 0\cdot 199$$
$$x = 1\cdot 61 \quad \text{or} \quad x = -1\cdot 61 .$$

Exercises 8.3

Solve the following equations for $x \in \mathbb{R}$. Give your answers correct to two decimal places.

1. $x^6 = 80$ **2.** $x^{5\cdot2} = 11\cdot7$ **3.** $(x+3)^5 = 21\cdot8$

4. $(x+5)^7 = 98$ **5.** $(2x-1)^{1\cdot3} = 273$ **6.** $(3x+2)^{4\cdot6} = 823\cdot6$

Solve the following equations for $x \in \mathbb{R}$.

7. $2^x = 32$ **8.** $3^x = \dfrac{1}{27}$ **9.** $2^{3x-1} = \dfrac{1}{64}$

10. $3^{2x+1} = \sqrt{27}$ **11.** $9^{x+2} = \dfrac{1}{27^{2x+5}}$ **12.** $4^{2x-1} = \sqrt{\dfrac{2}{8^{x+3}}}$

Without using a calculator, evaluate each of the following logs.

13. $\log_4 16$ **14.** $\log_3 \sqrt{27}$ **15.** $\log_4 128$ **16.** $\log_{16} \sqrt{128}$

Solve the following equations for $x \in \mathbb{R}$. Where appropriate, give your answers correct to two decimal places.

17. $2\cdot6^x = 9\cdot47$ **18.** $5\cdot18^x = 167$ **19.** $0\cdot92^x = 0\cdot05$

20. $3\cdot82^{2x+1} = 39\cdot7$ **21.** $1\cdot47^{3x-1} = 12\cdot8$ **22.** $10\cdot4^{2-x} = 43\cdot7$

23. $e^{2x+1} = 65\cdot9$ **24.** $e^{3x+4} = 87\cdot23$ **25.** $8\cdot9e^{3x+7} = 241\cdot87$

26. Solve the equation
$$7(2^{x+5}) = 1\cdot4(2^{2x+1})$$
for $x \in \mathbb{R}$. Give your answer correct to two decimal places.

27. Solve the equation
$$1\cdot3\times10^4 \times e^{3t-1} = 6\cdot5\times10^5 \times e^{t+3}$$
for $t \in \mathbb{R}$. Give your answer correct to two decimal places.

28. Solve the equation
$$8\cdot9(e^{4x+7}) = 32\cdot5(e^{3x-1})$$
for $x \in \mathbb{R}$. Give your answer correct to two decimal places.

29. Solve the equation
$$3\cdot2(2\cdot56)^x = 18\cdot9(1\cdot73)^{x+2}$$
for $x \in \mathbb{R}$. Give your answer correct to two decimal places.

30. Solve the equation

$$2^{2x} - 20(2^x) + 64 = 0\,,$$

for $x \in \mathbb{R}$.

31. Solve the equation

$$3^{2x+5} - 4(3^{x+2}) + 1 = 0$$

for $x \in \mathbb{R}$.

32. Solve the equation

$$3^{2x+1} - 28(3^x) + 9 = 0\,,$$

for $x \in \mathbb{R}$.

33. Solve the equation

$$\frac{2}{e^x} = e^x - 1$$

for $x \in \mathbb{R}$, correct to two decimal places.

8.4 Log Equations

1. **Definition of a Log Equation**
   A log equation is an equation containing logs. To solve such equations, we usually need to eliminate the logs from the equation.

2. **Solving Log Equations**
   To eliminate logs from an equation, we can write the equation in one or other of the following two formats.

   * **Single log equal to a number:**
   $$\log_a X = Y$$
   Thus $a^Y = X$

   * **Single log equal to a single log to the same base:**
   $$\log_a X = \log_a Y$$
   Thus $X = Y$.

3. **Checking Answers**
   As long as a is a positive number, $\log_a x$ is **only defined** if $x > 0$. Thus when we solve a log equation, any answer must be checked to make sure that we are not taking the log of a negative number or zero. Check each answer in the original equation.

Example 1
Solve the equation
$$\log_6(x-3)+\log_6(x+2)=1,$$
for $x \in \mathbb{R}$.
Solution
(We will write the equation as a single log equal to a number.)
$$\log_6(x-3)+\log_6(x+2)=1$$
$$\log_6(x-3)(x+2)=1$$
$$\log_6(x^2-x-6)=1$$
$$x^2-x-6=6^1$$
$$x^2-x-12=0$$
$$(x-4)(x+3)=0$$
$$x-4=0 \quad \text{or} \quad x+3=0$$
$$x=4 \quad \text{or} \quad x=-3$$

Check: $x=4$ OK

$x=-3$ Not OK ($\log_6(-6)$ is not defined.)

Answer: $x=4$.

Example 2

Solve the equation

$$\log_4(3x+1)=\log_2(x-1),$$

for $x\in\mathbb{R}$.

Solution

(We need to write both sides with the same base, say 2.)

$$\log_4(3x+1)=\frac{\log_2(3x+1)}{\log_2 4} \quad \textit{... Change of Base Law}$$

$$=\frac{\log_2(3x+1)}{2}$$

Then the equation is

$$\frac{\log_2(3x+1)}{2}=\log_2(x-1)$$

$$\log_2(3x+1)=2\log_2(x-1)$$

$$\log_2(3x+1)=\log_2(x-1)^2 \quad \textit{... Law 3}$$

$$3x+1=x^2-2x+1$$

$$0=x^2-5x$$

$$x(x-5)=0$$

$$x=0 \text{ or } x-5=0$$

$$x=0 \text{ or } x=5$$

Check: $x=0$ Not OK (The given equation contains $\log_2(-1)$.)

$x=5$ OK

Ans: $x=5$.

Exercises 8.4

Solve the following equations for $x\in\mathbb{R}$.

1. $\log_4 x+\log_4 2=\log_4 28$

2. $\log_3(2x-1)-\log_3(x-4)=2$

3. $\log_2(x-3)-2\log_2(x+3)=-5$

4. $\log_2(5x+1)=2\log_2(x+1)$

5. $\log_3(x+4)=\log_3(x-2)+\log_3(2x-7)$

6. $\log_5(x+3)-\log_5(2x-3)=\log_5(2x+1)$

7. **(i)** Show that $\log_4(5x+1)=\frac{1}{2}\log_2(5x+1)$.

(ii) Hence, or otherwise, solve the equation

$$\log_4(5x+1)=\log_2(x+1)$$

for $x\in\mathbb{R}$.

8. **(i)** Show that $\log_9(4x-15)=\frac{1}{2}\log_3(4x-15)$.

(ii) Hence, or otherwise, solve the equation

$$\log_3(x+3)-\log_9(4x-15)=1$$

for $x\in\mathbb{R}$.

9. Solve the simultaneous equations:

$$\log_2 x-\frac{1}{2}\log_2 y=4$$

$$\log_2(x-2y)=5.$$

8.5 Exponential and Log Functions and Graphs

A Exponential Functions and Relationships

1. Definition of an Exponential Function

* An exponential function is a function of the form

$f : x \to a(b)^x$, where $a, b \in \mathbb{R}, b > 0$.

* For example, $f(x) = 4(3)^x$ and $f(x) = 1\cdot 86(2\cdot 03)^x$ are both exponential functions.

* There are many real-life examples of exponential functions, e.g. the growth of an investment under compound interest, the decay of a radioactive substance.

2. Exponential Relationship from a Data Table

The following table gives corresponding values for two variables *x* and *y*.

x	0	1	2	3	4
y	150	165	$181\cdot 5$	$199\cdot 65$	$219\cdot 615$

* As the changes in the independent variable, *x*, are fixed, we can check if there is an exponential relationship between *x* and *y* by calculating the ratios of successive *y* values. If these are constant, then there is an exponential relationship between *x* and *y*, i.e. *y* is an exponential function of *x*.

* Checking:

(i) $\frac{165}{150} = 1\cdot 1$, (ii) $\frac{181\cdot 5}{165} = 1\cdot 1$, (iii) $\frac{199\cdot 65}{181\cdot 5} = 1\cdot 1$, (iv) $\frac{219\cdot 615}{199\cdot 65} = 1\cdot 1$.

As these ratios are constant, *y* is an exponential function of *x*.

Definition of an Exponential Relationship
The data in a data table represents an exponential relationship between the variables if
(i) for fixed changes in the independent variable,
(ii) the ratios of successive terms are constant.

3. Finding the Equation of an Exponential Relationship from a Data Table

* If there is an exponential relationship between *x* and *y*, we can write

  $y = a(b)^x$.

* We can then use two data pairs from the table to evaluate *a* and *b*.

Example 1

The table below gives corresponding values for the variables *x* and *y*.

x	0	1	2	3	4
y	150	165	$181\cdot5$	$199\cdot65$	$219\cdot615$

If there is an exponential relationship between *x* and *y*, find the values of the constants *a* and *b* if

$y = a(b)^x$.

Solution

[1] $x = 0$ when $y = 150$.

$150 = a(b)^0$

$150 = a$ ... as $b^0 = 1$

[2] $x = 1$ when $y = 165$.

$165 = 150(b)^1$

$165 = 150b$

$b = 1\cdot1$

Thus $y = 150(1\cdot1)^x$.

(Note that *b*, the base of the exponential function, is always the same as the constant ratio, as long as the differences in the *x* values is 1.)

B Natural Exponential Functions and Relationships

1. Natural Exponential Function

* The natural exponential function, or sometimes simply ‘the’ exponential function, is

  $f(x) = e^x$,

  where *e* is approximately $2\cdot718$.

* More generally, natural exponential functions can be written in the form
$$f(x) = ae^{bx},$$
where a and b are constants.

* Natural exponential functions, i.e. functions of the form
$$f(x) = ae^{bx}$$
are often preferred to exponential functions with other bases, i.e. functions of the form
$$f(x) = a(b)^x.$$
This is because of the suitability of the natural exponential functions for further work.

2. Finding the Equation of a Natural Exponential Function

If we are given data, e.g. in the form of a table, then by forming and solving two equations, the values of the constants a and b in
$$f(x) = ae^{bx}$$
can be found.

Example 2

The table below gives some values of y for the corresponding values of x.

x	0	1	2	3	4
y		$860 \cdot 7$			$548 \cdot 81$

(i) If $y = ae^{bx}$, use the values in the table to find the values of the constants a and b. Give a correct to the nearest integer and give b correct to two decimal places.

(ii) Hence complete the table above.

Solution

(i) $y = ae^{bx}$

$\underline{x = 1,\ y = 860 \cdot 7}$:

$$860 \cdot 7 = ae^{b(1)}$$
$$ae^{b} = 860 \cdot 7 \qquad \textbf{... 1}$$

$\underline{x = 4,\ y = 548 \cdot 81}$:

$$548 \cdot 81 = ae^{b(4)}$$
$$ae^{4b} = 548 \cdot 81 \qquad \textbf{... 2}$$

Dividing **2** by **1**:

$$\frac{ae^{4b}}{ae^{b}}=\frac{548\cdot 81}{860\cdot 7}$$

$$e^{3b}=0\cdot 637632$$
$$3b=\ln 0\cdot 637632$$
$$3b=-0\cdot 4499$$
$$b=-0\cdot 15$$

Then

1: $ae^{-0\cdot 15}=860\cdot 7$

$$a=\frac{860\cdot 7}{e^{-0\cdot 15}}$$
$$a=1000\,.$$

Thus

$$y=1000e^{-0\cdot 15x}\,.$$

(ii) The table is completed below. For example, when $x=2$,

$$y=1000e^{-0\cdot 15(2)}=740\cdot 81\,.$$

x	0	1	2	3	4
y	1000	$860\cdot 7$	$740\cdot 81$	$637\cdot 62$	$548\cdot 81$

3. Which Exponential Equation to Use

* If we are told that *y* is an exponential function of *x*, we can use either

$$y=a(b)^x$$

or $$y=ae^{bx},$$

unless we are instructed to use one in particular.

* You should note that the value of the constant *b* will depend on which form of the equation you choose.

C Exponential Graphs: Growth and Decay

1. Exponential Graphs from a Table of Values

* Like other functions, we can construct an exponential graph by forming a table of values.
* For example, consider the function

$$f(x)=2^x$$

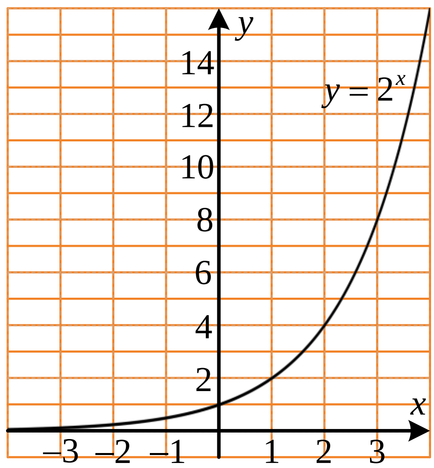
and the corresponding graph

$$y=2^x\,.$$

* We can construct a table of values for this function and hence plot the graph.

x	-4	-3	-2	-1	0	1	2	3	4
$y=2^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

* The key features of this curve are as follows.
(i) The curve lies completely above the x-axis.
(ii) On the left, the curve gets closer and closer to the x-axis. The x-axis is called an **asymptote** of the curve. An asymptote of a curve is a line which approximates the curve as the curve tends to infinity.
(iii) On the right, the curve increases more and more rapidly as x increases.



2. Definition of Exponential Growth

* y is said to be growing exponentially with x if y is an exponential function of x and the values of y increase as the values of x increase.

* The equation of exponential growth can be written (for $a>0$):
(i) $y=a(b)^x$, where $b>1$, or
(ii) $y=ae^{bx}$, where $b>0$.

3. Exponential Growth Graphs

* Some examples of exponential growth graphs are shown below.

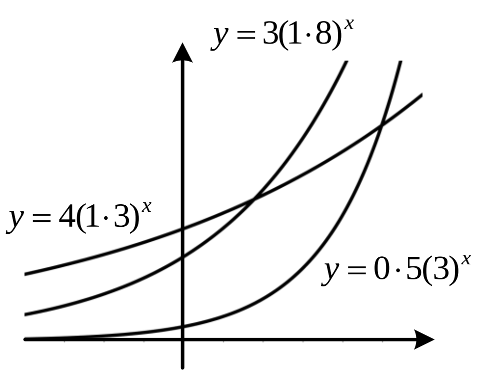


Fig. 1

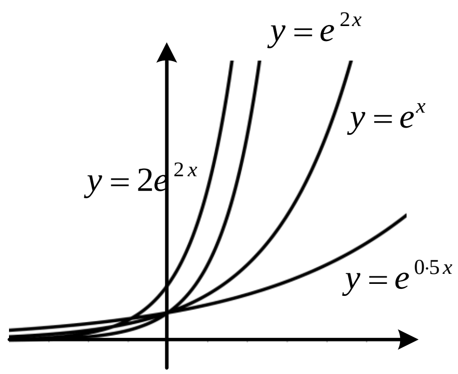


Fig. 2

* Fig. 1 shows graphs with equations of the form
$$y = a(b)^x, \qquad \text{with } a > 0 \text{ and } b > 1.$$

Notice that:
(i) The larger the base, b, the more steeply the graph rises on the right.
(ii) The graph intersects the y-axis at the point $(0, a)$.

* Fig. 2 shows graphs with equations of the form
$$y = a e^{bx}, \qquad \text{with } a > 0 \text{ and } b > 0.$$

Notice that:
(i) The larger the value of b, the more steeply the graph rises on the right.
(ii) The graph intersects the y-axis at the point $(0, a)$.

4. Definition of Exponential Decay

* y is said to be decaying exponentially with x if y is an exponential function of x and the values of y decrease as the values of x increase.

* The equation of exponential decay can be written (for $a > 0$):
(i) $y = a(b)^x$, where $0 < b < 1$, or
(ii) $y = a e^{bx}$, where $b < 0$, or
(iii) $y = a e^{-bx}$, where $b > 0$.

5. Exponential Decay Graphs

* Some examples of exponential decay graphs are shown below.

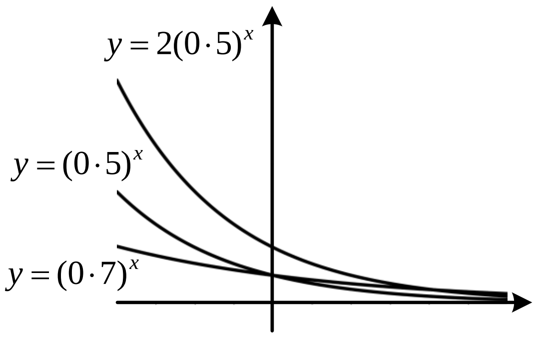


Fig. 1

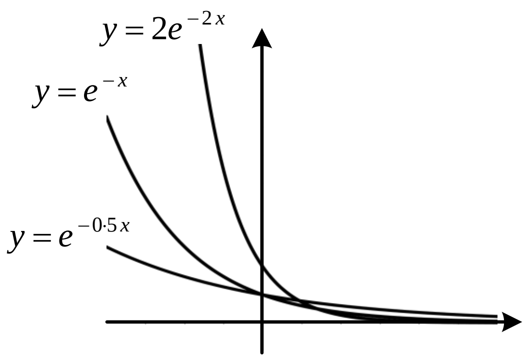


Fig. 2

* Fig. 1 shows graphs with equations of the form

  $y = a(b)^x$, with $a > 0$ and $0 < b < 1$.

  Notice that:
  (i) The smaller the base, b, the more steeply the graph falls on the right.
  (ii) The graph intersects the y-axis at the point $(0, a)$.

* Fig. 2 shows graphs with equations of the form

  $y = a e^{bx}$, with $a > 0$ and $b < 0$, or

  $y = a e^{-bx}$, with $a > 0$ and $b > 0$.

  Notice that:
  (i) The more negative the index, the more steeply the graph falls on the right.
  (ii) The graph intersects the y-axis at the point $(0, a)$.

D Logarithmic Functions and Graphs

1. Logarithmic Functions

* Function such as

  $f(x) = \log_a x$

  or $f(x) = \ln x$

  are called logarithmic (log) functions.
* These functions are only defined for $x > 0$.

2. Constructing Log Graphs

We can construct a log graph by
(i) making a table of values, and
(ii) using our knowledge of the shape of a log graph, as outlined below.

3. Exponential and Log Functions are Inverses

* The functions $y = a^x$ and $y = \log_a x$ are inverse functions of each other.
* To demonstrate this, we show that performing one after the other returns us to x.
  (i) $\log_a a^x = x \log_a a = x(1) = x$
  (ii) To find $a^{\log_a x}$, let

  $y = a^{\log_a x}$

  Then $\log_a y = \log_a x$

  $y = x$

  Thus $a^{\log_a x} = x$.

4. Graph of Log as Inverse of Exponential

* As discussed above, the functions

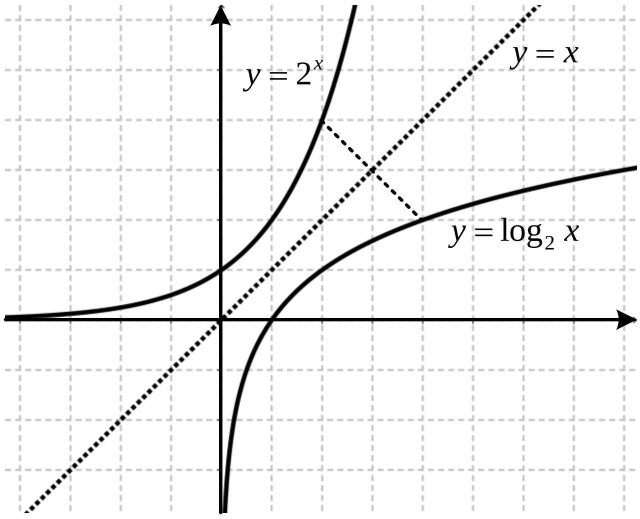
  $f(x) = 2^x$

  and $g(x) = \log_2 x$

  are inverses of each other.

* Because of this, the graph of each is the reflection of the other in the line $y = x$.

* The two graphs, $y = 2^x$ and $y = \log_2 x$, are shown opposite.



5. Other Log Graphs

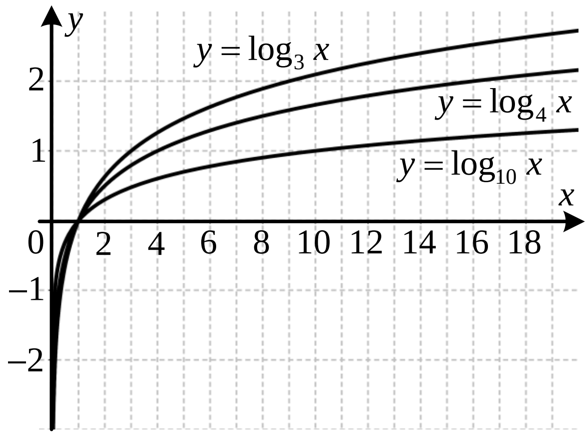
* The diagram opposite shows the three log graphs:

  $y = \log_3 x$,

  $y = \log_4 x$

  and $y = \log_{10} x$.

* Each of these graphs
  (i) is only defined for $x > 0$,
  (ii) crosses the x-axis at $(1,0)$,
  (iii) tends to $-\infty$ as x tends to 0 from the plus side (the y-axis is an asymptote)
  (iv) has a y co-ordinate of 1 when the x co-ordinate is equal to the base.

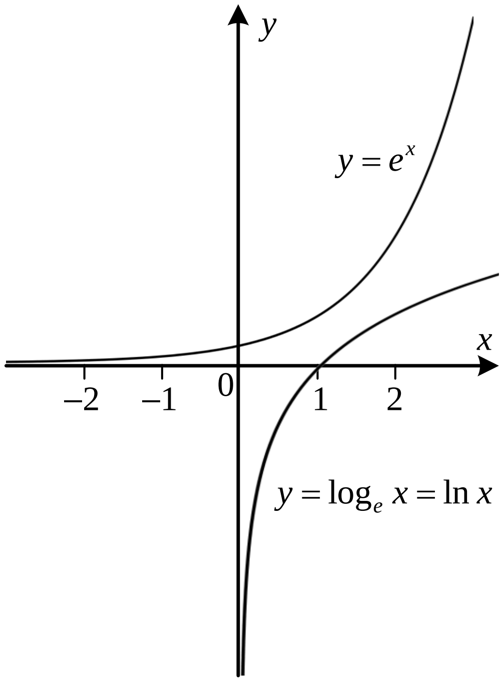


6. Natural Exponential and Natural Log are Inverses

Like other exponential and log functions with the same base, the natural exponential and natural log functions are inverses. Thus

(i) $\ln e^x = x$

(ii) $e^{\ln x} = x$.



7. Graphs

* As inverse functions, the graphs

$y = e^x$

and $y = \ln x$

are reflections of each other in the line $y = x$.

* These graphs are shown opposite.

E Solving Equations Using Intersecting Graphs

We can obtain approximate solutions of an equation containing exponential functions or log functions by finding the intersection of the types of graphs we have discussed previously.

Example 3

(i) If $f(x) = 2(1\cdot 5)^x$, construct a graph of $y = f(x)$ by completing the table:

x	-2	-1	0	1	2	3
y						

(ii) Using the same axes and the same scales, sketch a rough graph of the function

$g(x) \to x^2$.

(iii) Use your graph to estimate the solutions of the equation

$2(1\cdot 5)^x = x^2$.

(iv) What other methods, if any, could have been used to solve this equation?

Solution

(i) The function is

$$f(x) = 2(1\cdot 5)^x$$

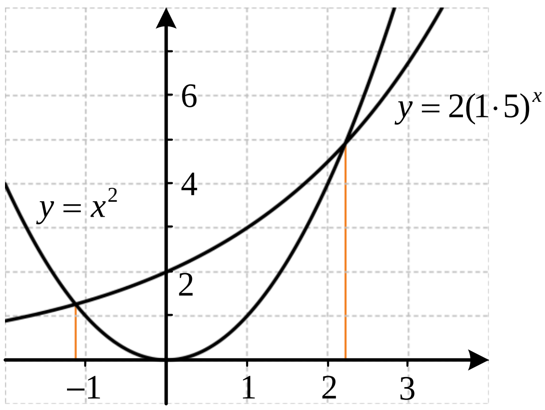
and the corresponding curve is

$$y = 2(1\cdot 5)^x .$$

Constructing a table of values:

x	-2	-1	0	1	2	3
y	$0\cdot 89$	$1\cdot 33$	2	3	$4\cdot 5$	$6\cdot 75$

The graph is shown below.



(ii) The function is

$$g(x) = x^2$$

and the corresponding curve is

$$y = x^2 .$$

This curve is also shown on the graph opposite.

(iii) The equation

$$2(1\cdot 5)^x = x^2$$

may be written

$$f(x) = g(x) .$$

The solutions of this equation are the x co-ordinates of the points of intersection of the curves $y = f(x)$ and $y = g(x)$. From the graph, the solutions are $x = -1\cdot 1$ and $x = 2\cdot 2$, correct to one decimal place.

(iv) We have no algebraic method on our course to solve the equation

$$2(1\cdot 5)^x = x^2 .$$

One other possible method is to estimate the solutions from tables of values, but the graphical method is undoubtedly the best available.

Exercises 8.5

1. The table below gives corresponding values for the variables *x* and *y*.

x	0	1	2	3	4
y	100	105	$110\cdot25$	$115\cdot7625$	$121\cdot550625$

(i) Use the data in the table to show that there is an exponential relationship between *x* and *y*.

(ii) If $y = a(b)^x$,
find the values of the constants *a* and *b*.

2. The table below gives corresponding values for the variables *x* and *y*.

x	0	2	4	6	8
y	500	480	$460\cdot8$	$442\cdot368$	$424\cdot67328$

(i) Use the data in the table to show that there is an exponential relationship between *x* and *y*.

(ii) If $y = a(b)^x$,
find the values of the constants *a* and *b*.

3. The table below gives some values of *y* for corresponding values of *x*.

x	0	1	2	3	4
y		$607\cdot43$		$1106\cdot8$	

(i) If $y = ae^{bx}$, use the values in the table to find the values of the constants *a* and *b*. Give *a* as an integer and *b* correct to one decimal place.

(ii) Complete the table of values above.

(iii) Find the value of *y* when $x = 8$. Give your answer correct to the nearest integer.

4. The function *f* is given by

$$f(x) = ae^{bx},$$

where *a* and *b* are constants.

(i) The curve $y = f(x)$ contains the points $(1, 53\cdot35)$ and $(2\cdot7, 190\cdot92)$. Find the value of *a*, correct to one decimal place, and the value of *b*, correct to two decimal places.

(ii) Find the value of *x* if $f(x) = 1000$. Give your answer correct to two decimal places.

5. If the curve

$$y = ae^{bx}$$

contains the points $(-1 \cdot 3, 129 \cdot 69)$ and $(2 \cdot 4, 61 \cdot 88)$, find the values of the constants *a* and *b*. Give *a* correct to the nearest integer and *b* correct to one decimal place.

6. $f : \mathbb{R} \to \mathbb{R}^+ : x \to 3(2)^x$.

(i) By completing the following table of values, construct a graph of the curve $y = f(x)$, for $-3 \le x \le 1$.

x	-3	-2	-1	0	1
$y = f(x)$					

(ii) Using the same axes and the same scales, construct a graph of the function

$$g : x \to x + 3, \quad \text{for } -3 \le x \le 1.$$

(iii) Use your graphs to find the solutions of the equation

$$3(2)^x = x + 3.$$

(iv) What other methods, if any, that we have seen can be used to solve this equation? Discuss.

7. $f : \mathbb{R} \to \mathbb{R}^+ : x \to \frac{2}{3}\left(\frac{3}{2}\right)^x$.

(i) By completing the following table of values, construct a graph of the curve $y = f(x)$, for $-2 \le x \le 4$.

x	-2	-1	0	1	2	3	4
$y = f(x)$							

(ii) Using the same axes and the same scales, construct a graph of the function

$$g : x \to \frac{1}{2}x + 1, \quad \text{for } -2 \le x \le 4.$$

(iii) Use your graphs to find the solutions of the equation

$$\frac{2}{3}\left(\frac{3}{2}\right)^x = \frac{1}{2}x + 1.$$

(iv) What other methods, if any, that we have seen can be used to solve this equation?

8. Two functions are

$$f:\mathbb{R}\to\mathbb{R}^+ : x \to 2(2\cdot 5)^x$$

and $g:\mathbb{R}\to\mathbb{R}^+ : x \to 3(0\cdot 5)^x$.

(i) By completing the following table of values, construct a graph of the curve $y=f(x)$, for $-2\le x\le 2$.

x	-2	-1	0	1	2
$y=f(x)$					

(ii) Using the same axes and the same scales, sketch a graph of $y=g(x)$, for $-2\le x\le 2$.

(iii) Use your graphs to find the solution of the equation

$$f(x)=g(x).$$

(iv) Check your answer to part (iii) by using algebra.

9. $f:\mathbb{R}^+\to\mathbb{R} : x\to \log_5 x$.

(i) Copy and complete the table:

x	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25
$y=f(x)$					

Hence construct the graph $y=f(x)$.

(ii) Using the same axes and the same scales, sketch the graph of

$$g: x\to x-2.$$

(iii) Use your graph to find the solutions of the equation

$$\log_5 x = x-2.$$

(iv) What other methods, if any, that we have seen can be used to solve this equation? Discuss.

8.6 Exponential Problems

1. Time in Real Life Problems

In real life problems, it is common practice to use the letter t to represent time, although other letters, e.g. x, can also be used to represent time.

2. Exponential Growth with Time

* In scientific work, it is very common to see a quantity, $Q(t)$, which grows exponentially with time, t, being expressed in the form

$$Q(t) = Ae^{bt},$$

where b is a positive constant.

* Note that A is the initial amount, i.e. the amount when $t = 0$.

3. Exponential Decay with Time

* Also in scientific work, it is very common to see a quantity, $Q(t)$, which decays exponentially with time, t, being expressed in one of the following forms:

(i) $Q(t) = Ae^{bt}$, where b is a negative constant, or

(ii) $Q(t) = Ae^{-bt}$, where b is a positive constant.

* Note that A is the initial amount, i.e. the amount when $t = 0$.

Example 1

In a laboratory experiment, a quantity, $Q(t)$, of a chemical was observed at various points in time, t. Time is measured in minutes from the initial observation. The table below gives the results.

t	0	1	2	3	4
$Q(t)$	$5\cdot9000$	$4\cdot8305$	$3\cdot9549$	$3\cdot2380$	$2\cdot6510$

Q follows a rule of the form $Q(t) = Ae^{-bt}$, where A and b are constants.

(i) Use two of the observations from your table to evaluate A and b. Verify your values by taking another observation from the table.

(ii) Hence find the value of Q after 10 minutes, i.e. $Q(10)$.

(iii) Estimate the time that it takes for the quantity of the chemical to reduce to 10% of the original amount. Give your answer in minutes, correct to three decimal places.

(iv) It is suggested that another model for Q is $Q(t) = Ap^t$. Express p in terms of b, correct to five decimal places.

Solution

(i) $Q(0)=5\cdot 9$: $\quad Ae^{-b(0)}=5\cdot 9$

$Ae^{0}=5\cdot 9$

$A=5\cdot 9$

$Q(1)=4\cdot 8305$: $\quad 5\cdot 9e^{-b(1)}=4\cdot 8305$

$e^{-b}=\dfrac{4\cdot 8305}{5\cdot 9}$

$e^{-b}=0\cdot 8187288$

$-b=\ln 0\cdot 8187288$

$-b=-0\cdot 2$

$b=0\cdot 2$

Thus $\quad Q(t)=5\cdot 9e^{-0\cdot 2t}$

To check:

$$Q(4)=5\cdot 9e^{-0\cdot 2(4)}=5\cdot 9e^{-0\cdot 8}=2\cdot 6510$$

Correct.

(ii) $Q(10)=5\cdot 9e^{-0\cdot 2(10)}$

$=5\cdot 9e^{-2}$

$=0\cdot 7895$

(iii) 10% of the original amount is $0\cdot 59$. Thus

$Q(t)=0\cdot 59$

$5\cdot 9e^{-0\cdot 2t}=0\cdot 59$

$e^{-0\cdot 2t}=0\cdot 1$

$-0\cdot 2t=\ln 0\cdot 1$

$-0\cdot 2t=-2\cdot 302585$

$t=11\cdot 513$ minutes.

(iv) $Q(t)=5\cdot 9e^{-0\cdot 2t}$

$Q(t)=5\cdot 9\left(e^{-0\cdot 2}\right)^{t}$

$Q(t)=5\cdot 9(0\cdot 81873)^{t}$

Thus $p=0\cdot 81873$, correct to five decimal places.

4. Exponential Growth: Quantity Growing at a Given Rate

* Suppose a quantity Q grows at a given rate, e.g. 4% per hour.
* This means that the quantity, at the end of any hour, is 4% greater than it was at the beginning of that hour.
* Thus, if P is the initial amount:
  (i) after 1 hour: $Q=P+0\cdot 04P=P(1\cdot 04)$ ... *multiply by* $1\cdot 04$
  (ii) after 2 hours: $Q=\left[P(1\cdot 04)\right](1\cdot 04)=P(1\cdot 04)^{2}$ ... *multiply by* $1\cdot 04$
  (iii) after t hours: $Q=P(1\cdot 04)^{t}$

* In general, if r is the growth rate, expressed as a decimal, then
$$Q = P(1+r)^t .$$

5. Exponential Decay: Quantity Decaying at a Given Rate

* Suppose a quantity Q decays at a given rate, e.g. 4% per hour.
* This means that the quantity, at the end of any hour, is 4% less than it was at the beginning of that hour.
* Thus, if P is the initial amount:
  (i) after 1 hour: $Q = P - 0\cdot 04P = P(1-0\cdot 04)$
  $= P(0\cdot 96)$ *... multiply by* $0\cdot 96$
  (ii) after 2 hours: $Q = \left[P(0\cdot 96)\right](0\cdot 96) = P(0\cdot 96)^2$ *... multiply by* $0\cdot 96$
  (iii) after t hours: $Q = P(0\cdot 96)^t = P(1-0\cdot 04)^t$
* In general, if r is the decay rate, expressed as a decimal, then
$$Q = P(1-r)^t .$$

Example 2

A biological sample starts with a size of P and grows at the rate of 3% each day after that. Let $Q(t)$ represent the size of the sample t days later.

(i) Determine what factor P is multiplied by after a period of one day.
(ii) Express $Q(t)$ in terms of t.
(iii) Find, correct to the nearest hour, how long it takes for the sample to double in size.
(iv) Find an expression for t in terms of n if $Q(t) = nP$.

Solution

(i) At the end of the first day, the size of the sample is
$$Q(1) = P + 3\% \text{ of } P$$
$$Q(1) = P + 0\cdot 03P$$
$$Q(1) = P(1+0\cdot 03)$$
$$Q(1) = P(1\cdot 03)$$
Thus the size of the sample is multiplied by the factor $1\cdot 03$ over a period of one day.

(ii) Hence
$$Q(2) = Q(1)\times(1\cdot 03)$$
$$= P(1\cdot 03)\times(1\cdot 03)$$
$$= P(1\cdot 03)^2$$
and $Q(t) = P(1\cdot 03)^t$

(iii) *(We want to find the value of t for which $Q(t)=2P$, i.e. when the sample size is twice its original size.)*

$$Q(t)=2P$$
$$P(1\cdot 03)^t=2P$$
$$(1\cdot 03)^t=2$$

Re-writing this equation in log form,

$$t=\log_{1\cdot 03}2$$
$$t=23\cdot 45 \text{ days}$$ ... *using the* $\log_{\square}\square$ *function on a calculator*
$$t=23 \text{ days and 11 hours,}$$
correct to the nearest hour.

(iv) Given

$$Q(t)=nP$$
$$P(1\cdot 03)^t=nP$$
$$(1\cdot 03)^t=n$$
$$t=\log_{1\cdot 03}n\,.$$

Example 3

A quantity of a radioactive substance starts with $2\cdot 4\times 10^{24}$ atoms. Each month after that 45% of the remaining substance decays. Write in the form $a\times 10^n$, where $1\le a<10$ and $n\in\mathbb{Z}$, the number of radioactive atoms remaining after one year.

Solution

Let $P(t)$ be the number of atoms remaining after *t* months. Then

$$P(t)=(2\cdot 4\times 10^{24})(1-0\cdot 45)^t$$
$$=(2\cdot 4\times 10^{24})(0\cdot 55)^t$$

After one year, $t=12$, and

$$P(12)=(2\cdot 4\times 10^{24})\times(0\cdot 55)^{12}$$
$$=(2\cdot 4\times 10^{24})\times(7\cdot 662\times 10^{-4})$$
$$=(2\cdot 4\times 7\cdot 662)\times 10^{24-4}$$
$$=18\cdot 3888\times 10^{20}$$
$$=1\cdot 83888\times 10^{21}$$

Exercises 8.6

1. The amount of a substance remaining in a solution reduces exponentially over time. An experiment measures the percentage of the substance remaining in the solution. Two minutes after the start, $67 \cdot 03\%$ of the substance remains, while five minutes after the start, $36 \cdot 79\%$ of the substance remains. Based on this, estimate the numbers of minutes from the start until 10% of the substance remains.

2. A laboratory obtains a 10 kg sample of a radioactive substance, which decays over time. The following measurements of weight, W, in grams, were obtained on a yearly basis. Let x represent the number of years from when the substance was obtained.

x / years	0	1	2	3	4
W / grams	10000	9000	8100	7290	6561

(i) Verify that W is an exponential function of x.
(ii) Express W in terms of x.
(iii) Use your expression for W to calculate the weight of radioactive substance present after $2 \cdot 75$ years.
(iv) Use logs to find the number of years it will take for the amount of radioactive substance to reduce to half the original amount. (This is called the 'half life' of the substance.) Give your answer in years to three decimal places.
(v) Determine the number of years, correct to three decimal places, that it will take for the weight of radioactive substance to reduce to 1 kg.

3. A chemical reaction starts with 1000 grams of reactant X, which then reduces exponentially as the reaction proceeds. The amount of X, in grams, present after t minutes is given by

$A(t) = Ce^{-bt}$, where C and b are constants.

The following table gives some values of A.

t	0	1	2	3	4
$A(t)$	1000				$367 \cdot 88$

(i) Find the value of C and the value of b. Hence complete the table above.
(ii) Find the value of t for which $A(t) = 50$.
(iii) Another reactant, Y, also starts with 1000 grams and reduces exponentially with time, so that the amount of Y remaining, $B(t)$, is given by

$B(t) = Ce^{-dt}$.

If after 4 minutes, there is less of Y remaining than of X, suggest a possible value for the constant d.
(iv) If after 15 minutes, the amount of Y remaining is half the amount of X remaining, find the value of d, correct to four decimal places.

4. The amount a company owes on a loan grows exponentially. The amount owed after t years, $A(t)$, is given by the following table.

t	0	1	2	3	4
$A(t)$		$5724 \cdot 46$	$5958 \cdot 08$		

A is given by a rule of the form $A(t) = Pe^{bt}$, where P and b are constants.

(i) Find the value of P and the value of b, and hence complete the table.

(ii) Find the number of years for the amount owed to double from its initial amount.

(iii) If $A(t)$ can also be written $A(t) = P(1+i)^t$, find the value of the constant i.

5. A bacteria colony grows by 4% per day.

(i) If the size at time $t = 0$ is 200, express $Q(t)$, the size after t days, in the form $Q(t) = Ae^{bt}$, where A and b are constants.

(ii) Find, correct to two decimal places, the time it takes for the colony to double in size.

(iii) Show that $\dfrac{Q(t+k)}{Q(t)}$ is independent of t.

6. In 2000, 100 grams of radium, a radioactive substance, were stored. This quantity decays exponentially over time. Let $Q(t) = Ae^{-bt}$ represent the amount of radium remaining t years later.

(i) The half-life of radium is 1602 years, i.e. it takes 1602 years for half of the original amount to decay. Find the values of the constants A and b.

(ii) Find the mass of radium that will remain in the year 4000, correct to two decimal places.

(iii) Find the mass of radium that will decay between the years 3000 and 4000.

7. Two bacteria colonies are growing exponentially with time. The size of one colony, t minutes after the start, is given by

$$P(t) = 8 \cdot 3 \times 10^6 \times e^{0 \cdot 15t}.$$

The size of the second colony, t minutes after the same start, is given by

$$Q(t) = 5 \cdot 2 \times 10^5 \times e^{0 \cdot 2t}.$$

Find the time, to the nearest minute, when the two colonies have the same size.

8. Each year 4% of the quantity of a radioactive substance present decays. The amount of radioactive substance present at the start was 500 grams. Let $y = f(x)$ be the amount of radioactive substance present after x years.

(i) Write down an expression for $f(1)$, the amount present after one year.

(ii) By writing down expressions for $f(2)$ and $f(3)$, find an expression for $f(x)$.

(iii) Find the number of years it takes for the amount of radioactive substance to reduce to 200 grams. Give your answer correct to the nearest year.

9. The number of bacteria present in a culture increases by 3% each hour. The number of bacteria present initially is N.

(i) If $f(t)$ represents the number of bacteria present after t hours, show that

$$f(t) = N(1 \cdot 03)^t .$$

(ii) Find the number of hours it takes for the number of bacteria present to increase to $2N$. Give your answer correct to the nearest hour.

(iii) If the number present after 6 hours is calculated to be 143286, find the value of N correct to the nearest unit.

10. A sum of money P, invested in a financial institution, grows by $2 \cdot 5\%$ each year after that. Let $A(t)$ be the amount to which the sum has grown after t years.

(i) Find an expression for $A(t)$.

(ii) Find t_1, the value of t for which the sum of money has grown by 50%.

(iii) Investigate if $A(2t_1) = 2P$.

(iv) If $A(t_1 + t_2) = 2P$, find the value of t_2.

11. A new machine costs €100,000. Its value depreciates by 8% per year after that. Let $f(t)$ be the value of the machine t years after being purchased.

(i) Express $f(t)$ in the form $a(b)^t$.

(ii) Find the number of years, correct to two decimal places, at which the value of the machine is half its original value.

(iii) The company that buys the machine plans on replacing it when it reaches 20% of its original value. Calculate the number of years the machine will have been in use before being replaced.

Revision Exercises 8

1. Solve the equation

$$2^x.2^{x+1} = 10$$

giving your answer

(i) in log form

(ii) correct to three decimal places.

2. Show that

$$\frac{8^n \times 2^{2n}}{4^{3n}} = \frac{1}{2^n}.$$

3. (i) Solve the equation

$$x + \frac{1}{x} = \frac{10}{3}.$$

(ii) Hence solve the equation

$$e^y + e^{-y} = \frac{10}{3}.$$

4. Solve the equation

$$2^{x+1} = 3^x,$$

giving your answer correct to three decimal places.

5. Siobhan is given a dose of radioactive medicine, which decays at the rate of 15% per hour after that. If the original dose is of 80 mg, and $f(x)$ mg represents the amount left in her body x hours after receiving the dose,

(i) calculate the amount left after 4 hours,

(ii) find the value of x when $f(x) = 20$,

(iii) the amount by which $f(x)$ decreases in the third hour.

6. $f(x) = Pe^{-kx}$ gives the amount of radioactive substance present x years after it starts to decay.

(i) What is the initial amount present, i.e. $f(0)$?

(ii) If it takes 10000 years for half the amount of radioactive substance to decay, express the value of k in scientific notation.

(iii) Express $f(1000)$ in terms of P.

7. A company wants to replace an existing machine in five years time. It estimates that the replacement will cost €500000 in five years time. It decides to set aside an amount €P now so that it will grow at the rate of 4% per year and will provide the finance to purchase the replacement. Calculate the value of P.

8. A bacteria colony grows at the rate of 8% per day.

(i) How many days will it take the colony to double in size?

(ii) How many days will it take the colony to treble in size?

(iii) If initially there are 8×10^7 bacteria present, write in scientific notation the number of bacteria present after 30 days.

9. A bacteria colony starts with a size of A and grows exponentially. It doubles in size in 10 minutes. Write the size, $Q(t)$, of the colony after t minutes in the form

$$Q(t) = Ae^{bt}$$

giving the value of b, correct to four decimal places.

Solutions to Exercises

Exercises 8.1

1. $2^5 \times 2^{-3} = 2^{5-3} = 2^2 = 4$

2. $\dfrac{3^5}{(3^{-2})^2} = \dfrac{3^5}{3^{-4}} = 3^{5-(-4)} = 3^9$

3. $\dfrac{\sqrt{8}\sqrt{32}}{2^{-3}} = \dfrac{\sqrt{256}}{2^{-3}} = \dfrac{16}{2^{-3}} = \dfrac{2^4}{2^{-3}} = 2^{4-(-3)} = 2^7 = 128$

4. $a^{2x+1}.a^{3x+1} = a^{(2x+1)+(3x+1)} = a^{5x+2}$

5. $\dfrac{(a^{x+1})^4}{a^{2-x}} = \dfrac{a^{4x+4}}{a^{2-x}} = a^{(4x+4)-(2-x)} = a^{5x+2}$

6. $\sqrt{\dfrac{b^{3x-1}}{b^{x+5}}} = \sqrt{b^{(3x-1)-(x+5)}}$

$= \sqrt{b^{2x-6}}$

$= \left(b^{2x-6}\right)^{\frac{1}{2}}$

$= b^{x-3}$

7. $(a^2b^3)^{-3} = (a^2)^{-3}(b^3)^{-3} = a^{-6}b^{-9}$

8. $\dfrac{xy^3}{(x^2y)^{-1}} = \dfrac{xy^3}{x^{-2}y^{-1}} = x^{1-(-2)}y^{3-(-1)} = x^3y^4$

9. $\left(\dfrac{p^2q}{q^4}\right)^2 = \left(\dfrac{p^2}{q^3}\right)^2 = \dfrac{(p^2)^2}{(q^3)^2} = \dfrac{p^4}{q^6}$

10. $5(4^{3n+1}) - 20(8^{2n}) = 5\left((2^2)^{3n+1}\right) - 20\left((2^3)^{2n}\right)$

$= 5\left(2^{6n+2}\right) - 20\left(2^{6n}\right)$

$= 5\left(2^{6n}.2^2\right) - 20\left(2^{6n}\right)$

$= 20(2^{6n}) - 20(2^{6n})$

$= 0$

11. $f(n) = 4(2^n)$

$f(n+k) = 4(2^{n+k})$

$= 4(2^n.2^k)$

$= 2^k\left[4(2^n)\right]$

$= 2^k f(n)$

12. $2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}=4\left(2^{\frac{1}{4}}\right)=2^2.2^{\frac{1}{4}}=2^{2+\frac{1}{4}}=2^{\frac{9}{4}}.$

13. (i) $e^{3x+2}.e^{4-2x}=e^{(3x+2)+(4-2x)}$

$=e^{x+6}$

(ii) $(e^{2x-1})^2.e^{5x+7}=e^{4x-2}.e^{5x+7}$

$=e^{9x+5}$

(iii) $\dfrac{e^{2x+5}}{e^{1-3x}}=e^{(2x+5)-(1-3x)}$

$=e^{2x+5-1+3x}$

$=e^{5x+4}$

(iv) $\dfrac{(e^{1-3x})^3}{e^{2-5x}}=\dfrac{e^{3-9x}}{e^{2-5x}}$

$=e^{(3-9x)-(2-5x)}$

$=e^{-4x+1}$

14. (i) $\dfrac{e^{x^2-2x+4}}{e^{1-3x}}=e^{(x^2-2x+4)-(1-3x)}$

$=e^{x^2+x+3}$

(ii) $\sqrt{\dfrac{e^{5x-1}}{e^{3-7x}}}=\sqrt{e^{(5x-1)-(3-7x)}}$

$=\sqrt{e^{12x-4}}$

$=\left(e^{12x-4}\right)^{\frac{1}{2}}$

$=e^{6x-2}$

(iii) $\dfrac{\sqrt{e^{3x+1}}}{(e^{2-x})^2}=\dfrac{(e^{3x+1})^{\frac{1}{2}}}{e^{4-2x}}$

$=\dfrac{e^{\frac{3}{2}x+\frac{1}{2}}}{e^{4-2x}}$

$=e^{\left(\frac{3}{2}x+\frac{1}{2}\right)-(4-2x)}$

$=e^{\frac{7}{2}x-\frac{7}{2}}$

(iv) $\dfrac{e^{x^2-x+4}}{\sqrt{e^{2x+6}}}=\dfrac{e^{x^2-x+4}}{(e^{2x+6})^{\frac{1}{2}}}$

$=\dfrac{e^{x^2-x+4}}{e^{x+3}}$

$=e^{(x^2-x+4)-(x+3)}$

$=e^{x^2-2x+1}$

15. $\left(e^x+\frac{1}{e^x}\right)^2=(e^x)^2+2(e^x)\left(\frac{1}{e^x}\right)+\left(\frac{1}{e^x}\right)^2$

$$=e^{2x}+2+\frac{1}{e^{2x}}$$

$$=e^{2x}+e^{-2x}+2$$

16. $\frac{e^{2x}+e^{x+2}}{e^{x+1}}=\frac{e^{2x}}{e^{x+1}}+\frac{e^{x+2}}{e^{x+1}}$

$$=e^{(2x)-(x+1)}+e^{(x+2)-(x+1)}$$

$$=e^{x-1}+e$$

Exercises 8.2

1. (i) $3^5=243$

$\log_3 243=5$

(ii) $7^4=2401$

$\log_7 2401=4$

(iii) $1\cdot 2^6=2\cdot 985984$

$\log_{1\cdot 2} 2\cdot 985984=6$

(iv) $3\cdot 5^{4\cdot 2}=192\cdot 7905695$

$\log_{3\cdot 5} 192\cdot 7905695=4\cdot 2$

2. (i) $\log_5 125=3$

$5^3=125$

(ii) $\log_3 6561=8$

$3^8=6561$

(iii) $\log_{1\cdot 05} 1\cdot 795856326=12$

$1\cdot 05^{12}=1\cdot 795856326$

(iv) $\log_{2\cdot 1} 438\cdot 7308937=8\cdot 2$

$2\cdot 1^{8\cdot 2}=438\cdot 7308937$

3. $\log_2 x+\log_2 y=\log_2 xy$

4. $\log_3 a^2-\log_3 b=\log_3\frac{a^2}{b}$

5. $\log_3 a-2\log_3 b+3\log_3 c=\log_3 a-\log_3 b^2+\log_3 c^3$

$$=\log_3\frac{ac^3}{b^2}$$

6. $\log_4 x+\frac{1}{2}\log_4 y-3\log_4 z=\log_4 x+\log_4 y^{\frac{1}{2}}-\log_4 z^3$

$$=\log_4\frac{x\sqrt{y}}{z^3}$$

7. (i) $\log_2 a^3 = 3\log_2 a = 3x$

(ii) $\log_2 16a^2 = \log_2 16 + \log_2 a^2$

$= \log_2 2^4 + 2\log_2 a$

$= 4\log_2 2 + 2x = 4 + 2x$

(iii) $\log_2 \dfrac{a^4}{32} = \log_2 a^4 - \log_2 32$

$= 4\log_2 a - \log_2 2^5$

$= 4x - 5\log_2 2 = 4x - 5$

8. (i) $p = \log_{27} q$

$q = 27^p$

(ii) $\log_9 q = \log_9 27^p = p\log_9 27$

Let $\log_9 27 = x$

$9^x = 27$

$(3^2)^x = 3^3$

$3^{2x} = 3^3$

$2x = 3$

$x = \dfrac{3}{2}$

Then

$\log_9 q = \dfrac{3}{2}p$

(iii) $\log_q 9 = \dfrac{\log_9 9}{\log_9 q} = \dfrac{1}{\log_9 q} = \dfrac{1}{\frac{3}{2}p} = \dfrac{2}{3p}$.

9. (i) $\log_a 108 = \log_a (4 \times 27)$

$= \log_a 2^2 + \log_a 3^3$

$= 2\log_a 2 + 3\log_a 3$

$= 2s + 3t$

(ii) $\log_a 48a^3 = \log_a 48 + \log_a a^3$

$= \log_a (16 \times 3) + 3\log_a a$

$= \log_a 2^4 + \log_a 3 + 3$

$= 4\log_a 2 + t + 3$

$= 4s + t + 3$

(iii) $\log_a \dfrac{27}{32a^4} = \log_a 27 - \log_a 32a^4$

$= \log_a 3^3 - \log_a 2^5 - \log_a a^4$

$= 3\log_a 3 - 5\log_a 2 - 4\log_a a$

$= 3t - 5s - 4$

10. (i) $\log_a x = \log_a y$

$\log_a x - \log_a y = 0$

$\log_a \dfrac{x}{y} = 0$

$\dfrac{x}{y} = a^0 = 1$

$x = y$

(ii) $\log_4 x = \log_2 y$

$\log_4 x = \dfrac{\log_4 y}{\log_4 2}$

$\log_4 x = \dfrac{\log_4 y}{\frac{1}{2}} = 2\log_4 y$

$\log_4 x = \log_4 y^2$

$x = y^2$

11. (i) $e^{2t} = p$

$\log_e p = 2t$

$\ln p = 2t$

(ii) $e^{1+3x} = 4y$

$\log_e 4y = 1 + 3x$

$\ln 4y = 1 + 3x$

(iii) $e^{x^2 - x} = 3\cdot 2$

$\log_e 3\cdot 2 = x^2 - x$

$\ln 3\cdot 2 = x^2 - x$

12. (i) $\ln(x^2 - 3x) = y$

$e^y = x^2 - 3x$

(ii) $\ln(3x + 7) = 2\cdot 8$

$e^{2\cdot 8} = 3x + 7$

(iii) $\ln(5t - 1) = 0\cdot 15x$

$e^{0\cdot 15x} = 5t - 1$

(iv) $\ln(2a + b) = 2\cdot 19$

$e^{2\cdot 19} = 2a + b$

13. $\log_e(x^2 e^{-x}) = \log_e x^2 + \log_e e^{-x}$

$= 2\log_e x + (-x)\log_e e$

$= 2\log_e x - x$

14. $\log_e e^{\sin x + 2} = (\sin x + 2)\log_e e$

$= \sin x + 2$

15. $\log_e(x^4 e^{\cos x}) = \log_e x^4 + \log_e e^{\cos x}$

$= 4\log_e x + (\cos x)\log_e e$

$= 4\log_e x + \cos x$

16. $\ln\left(x^2\sqrt{x+1}\right) = \ln x^2 + \ln(x+1)^{\frac{1}{2}}$

$= 2\ln x + \dfrac{1}{2}\ln(x+1)$

17. $\ln(x^3 e^{\sin x}) = \ln x^3 + \ln e^{\sin x}$

$= 3\ln x + (\sin x)\ln e$

$= 3\ln x + \sin x$

18. $\ln\left(\dfrac{\sqrt{x}}{e^{4x}}\right) = \ln\sqrt{x} - \ln e^{4x}$

$= \ln x^{\frac{1}{2}} - (4x)\ln e$

$= \dfrac{1}{2}\ln x - 4x$

Exercises 8.3

1. $x^6 = 80$

$x = 80^{\frac{1}{6}} = 2\cdot 08$

2. $x^{5\cdot 2} = 11\cdot 7$

$x = 11\cdot 7^{\frac{1}{5\cdot 2}} = 1\cdot 60$

3. $(x+3)^5 = 21\cdot 8$

$x + 3 = 21\cdot 8^{\frac{1}{5}} = 1\cdot 85$

$x = -1\cdot 15$

4. $(x+5)^7 = 98$

$x + 5 = 98^{\frac{1}{7}} = 1\cdot 93$

$x = -3\cdot 07$

5. $(2x-1)^{1\cdot 3} = 273$

$2x - 1 = 273^{\frac{1}{1\cdot 3}} = 74\cdot 812$

$2x = 75\cdot 812$

$x = 37\cdot 91$

6. $(3x+2)^{4\cdot 6} = 823\cdot 6$

$3x + 2 = 823\cdot 6^{\frac{1}{4\cdot 6}} = 4\cdot 303$

$3x = 2\cdot 303$
$x = 0\cdot 77$

7. $2^x = 32$
$2^x = 2^5$
$x = 5$

8. $3^x = \dfrac{1}{27}$
$3^x = \dfrac{1}{3^3}$
$3^x = 3^{-3}$
$x = -3$

9. $2^{3x-1} = \dfrac{1}{64}$
$2^{3x-1} = \dfrac{1}{2^6}$
$2^{3x-1} = 2^{-6}$
$3x - 1 = -6$
$3x = -5$
$x = -\dfrac{5}{3}$

10. $3^{2x+1} = \sqrt{27}$
$3^{2x+1} = (3^3)^{\frac{1}{2}}$
$3^{2x+1} = 3^{\frac{3}{2}}$
$2x + 1 = \dfrac{3}{2}$
$2x = \dfrac{1}{2}$
$x = \dfrac{1}{4}$

11. $9^{x+2} = \dfrac{1}{27^{2x+5}}$
$(3^2)^{x+2} = \dfrac{1}{(3^3)^{2x+5}}$
$3^{2x+4} = \dfrac{1}{3^{6x+15}}$
$3^{2x+4} = 3^{-6x-15}$
$2x + 4 = -6x - 15$
$8x = -19$
$x = -\dfrac{19}{8}$

12. $4^{2x-1}=\sqrt{\dfrac{2}{(2^3)^{x+3}}}$

$(2^2)^{2x-1}=\sqrt{\dfrac{2}{2^{3x+9}}}$

$2^{4x-2}=\sqrt{2^{1-(3x+9)}}$

$2^{4x-2}=(2^{-3x-8})^{\frac{1}{2}}$

$2^{4x-2}=2^{\frac{1}{2}(-3x-8)}$

$4x-2=\dfrac{1}{2}(-3x-8)$

$8x-4=-3x-8$

$11x=-4$

$x=-\dfrac{4}{11}$

13. $\log_4 16=x$

$4^x=16$

$4^x=4^2$

$x=2$

14. $\log_3\sqrt{27}=x$

$3^x=(3^3)^{\frac{1}{2}}$

$3^x=3^{\frac{3}{2}}$

$x=\dfrac{3}{2}$

15. $\log_4 128=x$

$(2^2)^x=2^7$

$2^{2x}=2^7$

$2x=7$

$x=\dfrac{7}{2}$

16. $\log_{16}\sqrt{128}=x$

$(2^4)^x=(2^7)^{\frac{1}{2}}$

$2^{4x}=2^{\frac{7}{2}}$

$4x=\dfrac{7}{2}$

$x=\dfrac{7}{8}$

17. $2\cdot 6^x = 9\cdot 47$

$x = \log_{2\cdot 6} 9\cdot 47$

$x = 2\cdot 35$

18. $5\cdot 18^x = 167$

$x = \log_{5\cdot 18} 167$

$x = 3\cdot 11$

19. $0\cdot 92^x = 0\cdot 05$

$\log_{0\cdot 92} 0\cdot 05 = x$

$x = 35\cdot 92$

20. $3\cdot 82^{2x+1} = 39\cdot 7$

$\log_{3\cdot 82} 39\cdot 7 = 2x+1$

$2\cdot 747 = 2x+1$

$2x = 1\cdot 747$

$x = 0\cdot 87$

21. $1\cdot 47^{3x-1} = 12\cdot 8$

$\log_{1\cdot 47} 12\cdot 8 = 3x-1$

$6\cdot 617 = 3x-1$

$3x = 7\cdot 617$

$x = 2\cdot 54$

22. $10\cdot 4^{2-x} = 43\cdot 7$

$\log_{10\cdot 4} 43\cdot 7 = 2-x$

$1\cdot 613 = 2-x$

$x = 0\cdot 39$

23. $e^{2x+1} = 65\cdot 9$

$\ln 65\cdot 9 = 2x+1$

$4\cdot 188 = 2x+1$

$2x = 3\cdot 188$

$x = 1\cdot 59$

24. $e^{3x+4} = 87\cdot 23$

$\ln 87\cdot 23 = 3x+4$

$4\cdot 469 = 3x+4$

$3x = 0\cdot 469$

$x = 0\cdot 16$

25. $8\cdot 9e^{3x+7} = 241\cdot 87$

$e^{3x+7} = 27\cdot 1764$

$\ln 27\cdot 1764 = 3x+7$

$3\cdot 302 = 3x+7$

$3x = -3\cdot 697$

$x = -1\cdot 23$

26. $7(2^{x+5}) = 1\cdot 4(2^{2x+1})$

$$\frac{7}{1\cdot 4} = \frac{2^{2x+1}}{2^{x+5}}$$

$$5=2^{x-4}$$
$$\log_2 5=x-4$$
$$2\cdot 32=x-4$$
$$x=6\cdot 32$$

27. $1\cdot 3\times 10^4\times e^{3t-1}=6\cdot 5\times 10^5\times e^{t+3}$

$$\frac{e^{3t-1}}{e^{t+3}}=\frac{6\cdot 5\times 10^5}{1\cdot 3\times 10^4}$$
$$e^{2t-4}=50$$
$$\ln 50=2t-4$$
$$3\cdot 912=2t-4$$
$$2t=7\cdot 912$$
$$t=3\cdot 96$$

28. $8\cdot 9(e^{4x+7})=32\cdot 5(e^{3x-1})$

$$\frac{e^{4x+7}}{e^{3x-1}}=\frac{32\cdot 5}{8\cdot 9}$$
$$e^{x+8}=3\cdot 652$$
$$\ln 3\cdot 652=x+8$$
$$1\cdot 295=x+8$$
$$x=-6\cdot 70$$

29. $3\cdot 2(2\cdot 56)^x=18\cdot 9(1\cdot 73)^{x+2}$

$$\ln\left[3\cdot 2(2\cdot 56)^x\right]=\ln\left[18\cdot 9(1\cdot 73)^{x+2}\right]$$
$$\ln 3\cdot 2+\ln 2\cdot 56^x=\ln 18\cdot 9+\ln 1\cdot 73^{x+2}$$
$$\ln 3\cdot 2+x\ln 2\cdot 56=\ln 18\cdot 9+(x+2)\ln 1\cdot 73$$
$$1\cdot 163+0\cdot 94x=2\cdot 939+0\cdot 548(x+2)$$
$$0\cdot 392x=2\cdot 872$$
$$x=7\cdot 33$$

30. Let $y=2^x$. Then

$$2^{2x}=(2^x)^2=y^2$$

The equation is

$$y^2-20y+64=0$$
$$(y-4)(y-16)=0$$
$$y=4 \quad \text{or} \quad y=16$$
$$2^x=2^2 \quad \text{or} \quad 2^x=2^4$$
$$x=2 \quad \text{or} \quad x=4$$

31. Let $y=3^x$. Then

$$3^{x+2}=3^x.3^2=9y$$
$$3^{2x+5}=(3^x)^2.3^5=243y^2$$

The equation is

$$243y^2 - 36y + 1 = 0$$
$$(27y-1)(9y-1) = 0$$
$$27y - 1 = 0 \quad \text{or} \quad 9y - 1 = 0$$
$$y = \frac{1}{27} \quad \text{or} \quad y = \frac{1}{9}$$
$$3^x = 3^{-3} \quad \text{or} \quad 3^x = 3^{-2}$$
$$x = -3 \quad \text{or} \quad x = -2$$

32. Let $y = 3^x$. Then

$$3^{2x+1} = 3(3^x)^2 = 3y^2$$

The equation is

$$3y^2 - 28y + 9 = 0$$
$$(3y-1)(y-9) = 0$$
$$3y - 1 = 0 \quad \text{or} \quad y - 9 = 0$$
$$y = \frac{1}{3} \quad \text{or} \quad y = 9$$
$$3^x = 3^{-1} \quad \text{or} \quad 3^x = 3^2$$
$$x = -1 \quad \text{or} \quad x = 2\,.$$

33. Let $y = e^x$. Then the equation can be written:

$$\frac{2}{y} = y - 1$$
$$2 = y^2 - y$$
$$y^2 - y - 2 = 0$$
$$(y-2)(y+1) = 0$$
$$y = 2 \quad \text{or} \quad y = -1$$
$$e^x = 2 \quad \text{or} \quad e^x = -1 \qquad \text{(not possible)}$$
$$x = \ln 2$$
$$x = 0\cdot 70\,.$$

Exercises 8.4

1. $\log_4 x + \log_4 2 = \log_4 28$

$$\log_4 2x = \log_4 28$$
$$2x = 28$$
$$x = 14$$

2. $\log_3(2x-1) - \log_3(x-4) = 2$

$$\log_3 \frac{2x-1}{x-4} = 2$$

$$\frac{2x-1}{x-4}=3^2=9$$
$$2x-1=9x-36$$
$$35=7x$$
$$x=5$$

3. $\log_2(x-3)-2\log_2(x+3)=-5$

$$\log_2(x-3)-\log_2(x+3)^2=-5$$
$$\log_2\frac{x-3}{(x+3)^2}=-5$$
$$\frac{x-3}{(x+3)^2}=2^{-5}=\frac{1}{32}$$
$$32(x-3)=x^2+6x+9$$
$$32x-96=x^2+6x+9$$
$$0=x^2-26x+105$$
$$(x-5)(x-21)=0$$
$$x-5=0 \quad \text{or} \quad x-21=0$$
$$x=5 \quad \text{or} \quad x=21$$

4. $\log_2(5x+1)=2\log_2(x+1)$

$$\log_2(5x+1)=\log_2(x+1)^2$$
$$5x+1=x^2+2x+1$$
$$0=x^2-3x$$
$$x(x-3)=0$$
$$x=0 \quad \text{or} \quad x-3=0$$
$$x=0 \quad \text{or} \quad x=3$$

(OK) (OK)

Ans: $x=0$ or $x=3$

5. $\log_3(x+4)=\log_3(x-2)+\log_3(2x-7)$

$$\log_3(x+4)=\log_3(x-2)(2x-7)$$
$$x+4=2x^2-11x+14$$
$$0=2x^2-12x+10$$
$$x^2-6x+5=0$$
$$(x-5)(x-1)=0$$
$$x-5=0 \quad \text{or} \quad x-1=0$$
$$x=5 \quad \text{or} \quad x=1$$

(OK) (not OK)

Ans: $x=5$

6. $\log_5(x+3)-\log_5(2x-3)=\log_5(2x+1)$

$$\log_5\frac{x+3}{2x-3}=\log_5(2x+1)$$
$$\frac{x+3}{2x-3}=2x+1$$

$x+3=(2x-3)(2x+1)$
$x+3=4x^2-4x-3$
$0=4x^2-5x-6$
$(x-2)(4x+3)=0$
$x-2=0$ or $4x+3=0$
$x=2$ or $x=-\frac{3}{4}$
(OK) (not OK)
Ans: $x=2$

7. (i) $\log_4(5x+1)=\frac{\log_2(5x+1)}{\log_2 4}=\frac{1}{2}\log_2(5x+1)$

(ii) $\log_4(5x+1)=\log_2(x+1)$
$\frac{1}{2}\log_2(5x+1)=\log_2(x+1)$
$\log_2(5x+1)=2\log_2(x+1)$
$\log_2(5x+1)=\log_2(x+1)^2$
$5x+1=x^2+2x+1$
$0=x^2-3x$
$x(x-3)=0$
$x=0$ or $x=3$

8. (i) $\log_9(4x-15)=\frac{\log_3(4x-15)}{\log_3 9}=\frac{1}{2}\log_3(4x-15)$

(ii) $\log_3(x+3)-\log_9(4x-15)=1$
$\log_3(x+3)-\frac{1}{2}\log_3(4x-15)=1$
$2\log_3(x+3)-\log_3(4x-15)=2$
$\log_3\frac{(x+3)^2}{4x-15}=2$
$\frac{x^2+6x+9}{4x-15}=3^2=9$
$x^2+6x+9=36x-135$
$x^2-30x+144=0$
$(x-6)(x-24)=0$
$x-6=0$ or $x-24=0$
$x=6$ or $x=24$

9. $\log_2 x-\frac{1}{2}\log_2 y=4$
$2\log_2 x-\log_2 y=8$
$\log_2 x^2-\log_2 y=8$

$\log_2 \frac{x^2}{y} = 8$

$\frac{x^2}{y} = 2^8$

$x^2 = 256y$ **... 1**

Also:

$\log_2(x - 2y) = 5$

$x - 2y = 2^5$

$x - 2y = 32$

$x = 2y + 32$ **... 2**

Then

$(2y + 32)^2 = 256y$

$4y^2 + 128y + 1024 = 256y$

$4y^2 - 128y + 1024 = 0$

$y^2 - 32y + 256 = 0$

$(y - 16)^2 = 0$

$y - 16 = 0$

$y = 16$

Then:

2: $x = 2(16) + 32$

$x = 64$

Thus $x = 64,\ y = 16$.

Exercises 8.5

1. (i) $\frac{105}{100} = 1\cdot 05,\ \frac{110\cdot 25}{105} = 1\cdot 05,\ \frac{115\cdot 7625}{110\cdot 25} = 1\cdot 05,\ \frac{121\cdot 550625}{115\cdot 7625} = 1\cdot 05$

As these ratios are constant, y is an exponential function of x.

(ii) $y = a(b)^x$

$x = 0,\ y = 100:$ $100 = a(b)^0$

$a = 100$

$x = 1,\ y = 105:$ $105 = 100(b)^1$

$b = 1\cdot 05$

Thus $y = 100(1\cdot 05)^x$.

2. (i) $\frac{480}{500} = 0\cdot 96,\ \frac{460\cdot 8}{480} = 0\cdot 96,\ \frac{442\cdot 368}{460\cdot 8} = 0\cdot 96,\ \frac{424\cdot 67328}{442\cdot 368} = 0\cdot 96$

As these ratios are constant, y is an exponential function of x.

(ii) $y = a(b)^x$

$x = 0,\ y = 500:$ $\quad 500 = a(b)^0$

$a = 500$

$x = 2,\ y = 480:$ $\quad 480 = 500(b)^2$

$b^2 = 0 \cdot 96$

$b = 0 \cdot 9798$

Thus $y = 500(0 \cdot 9798)^x$.

3. (i) $\underline{x = 1,\ y = 607 \cdot 43:}$

$607 \cdot 43 = ae^{b(1)}$

$ae^b = 607 \cdot 43$ **... 1**

$\underline{x = 3,\ y = 1106 \cdot 8:}$

$1106 \cdot 8 = ae^{b(3)}$

$ae^{3b} = 1106 \cdot 8$ **... 2**

Dividing **2** by **1**:

$$\frac{ae^{3b}}{ae^b} = \frac{1106 \cdot 8}{607 \cdot 43}$$

$e^{2b} = 1 \cdot 8221$

$2b = \ln 1 \cdot 8221$

$2b = 0 \cdot 6$

$b = 0 \cdot 3$

1: $ae^{0 \cdot 3} = 607 \cdot 43$

$$a = \frac{607 \cdot 43}{e^{0 \cdot 3}}$$

$a = 450$

Thus $y = 450e^{0 \cdot 3x}$

(ii) The table is completed below.

x	0	1	2	3	4
y	450	$607 \cdot 43$	$819 \cdot 95$	$1106 \cdot 8$	1494

(iii) $x = 8:$ $\quad y = 450e^{0 \cdot 3(8)}$

$y = 4960$

4. (i) $(1, 53 \cdot 35):$ $\quad 53 \cdot 35 = ae^{b(1)}$

$ae^b = 53 \cdot 35$ **... 1**

$(2 \cdot 7, 190 \cdot 92):$ $\quad 190 \cdot 92 = ae^{b(2 \cdot 7)}$

$ae^{2 \cdot 7b} = 190 \cdot 92$ **... 2**

Dividing **2** by **1**:

$$\frac{ae^{2 \cdot 7b}}{ae^b} = \frac{190 \cdot 92}{53 \cdot 35}$$

$e^{1\cdot 7b} = 3\cdot 5786$
$1\cdot 7b = \ln 3\cdot 5786$
$1\cdot 7b = 1\cdot 27498$
$b = 0\cdot 75$

1: $ae^{0\cdot 75} = 53\cdot 35$

$a = \dfrac{53\cdot 35}{e^{0\cdot 75}}$

$a = 25\cdot 2$

Thus $y = 25\cdot 2e^{0\cdot 75x}$

(ii) $f(x) = 1000$

$25\cdot 2e^{0\cdot 75x} = 1000$

$e^{0\cdot 75x} = 39\cdot 6825$
$0\cdot 75x = \ln 39\cdot 6825$
$0\cdot 75x = 3\cdot 6809$
$x = 4\cdot 91$

5. (i) $(-1\cdot 3, 129\cdot 69)$: $129\cdot 69 = ae^{b(-1\cdot 3)}$

$ae^{-1\cdot 3b} = 129\cdot 69$ ... **1**

$(2\cdot 4, 61\cdot 88)$: $61\cdot 88 = ae^{b(2\cdot 4)}$

$ae^{2\cdot 4b} = 61\cdot 88$ ... **2**

Dividing **2** by **1**:

$\dfrac{ae^{2\cdot 4b}}{ae^{-1\cdot 3b}} = \dfrac{61\cdot 88}{129\cdot 69}$

$e^{3\cdot 7b} = 0\cdot 4771$
$3\cdot 7b = \ln 0\cdot 4771$
$3\cdot 7b = -0\cdot 4771$
$b = -0\cdot 2$

1: $129\cdot 69 = ae^{-1\cdot 3(-0\cdot 2)}$

$ae^{0\cdot 26} = 129\cdot 69$

$a = \dfrac{129\cdot 69}{e^{0\cdot 26}}$

$a = 100$.

6. **(i)** $f(x) = 3(2)^x$

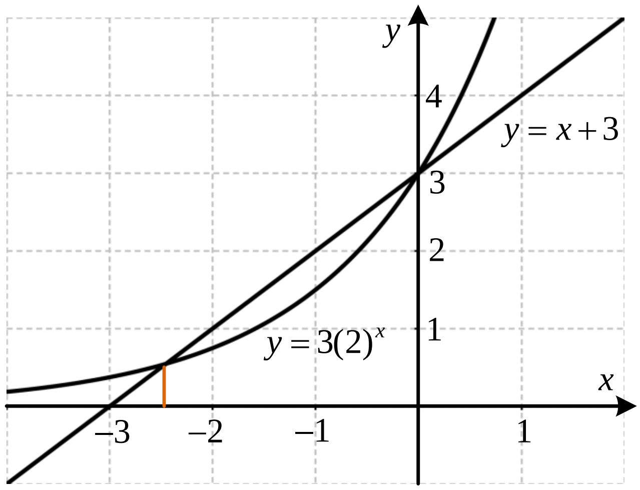
Table:

x	-3	-2	-1	0	1
$y = f(x)$	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	3	6

The graph is constructed below.

(ii) $g(x) = x + 3$

The linear graph $y = x + 3$ contains the points $(-3, 0)$ and $(0, 3)$, and is shown opposite.



(iii) From the graph, the solutions are approximately $-2 \cdot 5$ and 0.

(iv) We could compare values in the tables of values, but there is no algebraic method we have seen that can be used to solve this equation.

7. **(i)** $f(x) = \frac{2}{3}\left(\frac{3}{2}\right)^x$

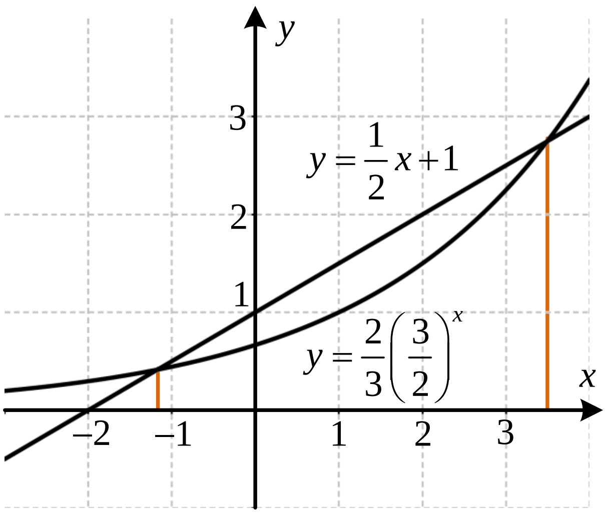
Table:

x	-2	-1	0	1	2	3	4
$y = f(x)$	$0 \cdot 3$	$0 \cdot 44$	$0 \cdot 67$	1	$1 \cdot 5$	$2 \cdot 25$	$3 \cdot 375$

The graph is constructed below.

(ii) $g(x) = \frac{1}{2}x + 1$

The linear graph $y = \frac{1}{2}x + 1$ contains the points $(0, 1)$ and $(2, 2)$, and is shown opposite.



(iii) From the graph, the solutions of the equation are approximately $-1 \cdot 2$ and $3 \cdot 4$.

(iv) We could compare values in the tables of values, but there is no algebraic method we have seen that can be used to solve this equation.

8. **(i)** $f(x)=2(2\cdot5)^x$

Table:

x	-2	-1	0	1	2
$y=f(x)$	$0\cdot32$	$0\cdot8$	2	5	$12\cdot5$

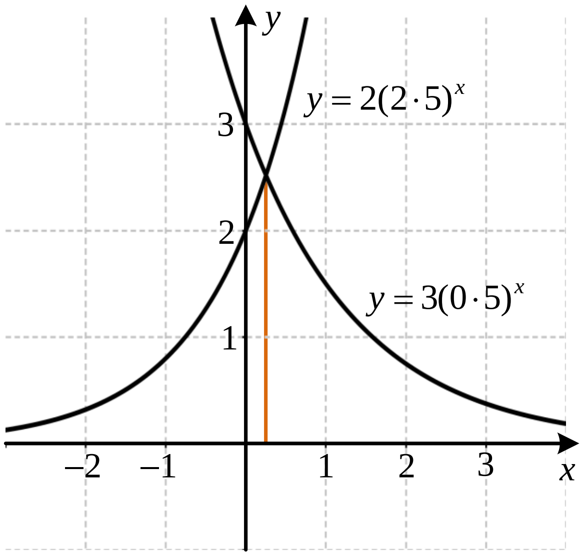
The graph is constructed below.

(ii) $g(x)=3(0\cdot5)^x$

Table:

x	-2	-1	0	1	2
$y=g(x)$	12	6	3	$1\cdot5$	$0\cdot75$

The graph is constructed below.



(iii) From the graph, the only solution of the equation is $0\cdot25$.

(iv) $f(x)=g(x)$

$$2(2\cdot5)^x=3(0\cdot5)^x$$

$$2\left(\frac{5}{2}\right)^x=3\left(\frac{1}{2}\right)^x$$

$$2\times\frac{5^x}{2^x}=3\times\frac{1}{2^x}$$

$$2\times5^x=3$$

$$5^x=1\cdot5$$

$$x=\log_5 1\cdot5$$

$$x=0\cdot2519$$

9. **(i)** $f(x)=\log_5 x$

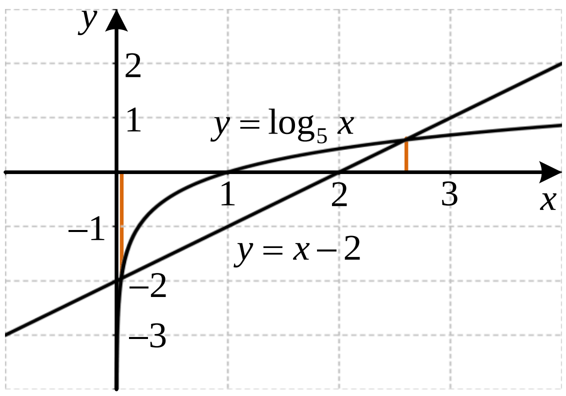
Table:

x	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25
$y=f(x)$	-2	-1	0	1	2

The graph is constructed below.

(ii) $g(x)=x-2$

The linear graph $y=x-2$ contains the points $(2,0)$ and $(0,-2)$, and is shown opposite.



(iii) From the graph, the solutions of the equation are approximately $0\cdot 1$ and $2\cdot 7$.

(iv) We could compare values in the tables of values, but there is no algebraic method we have seen that can be used to solve this equation.

Exercises 8.6

1. Let $A =$ percentage remaining after t minutes.

Let $A = a(b)^t$

$\underline{t = 2,\ A = 67\cdot 03}$:

$$67\cdot 03 = a(b)^2$$
$$ab^2 = 67\cdot 03 \qquad \textbf{... 1}$$

$\underline{t = 5,\ A = 36\cdot 79}$:

$$36\cdot 79 = a(b)^5$$
$$ab^5 = 36\cdot 79 \qquad \textbf{... 2}$$

Dividing **2** by **1**:

$$\frac{ab^5}{ab^2} = \frac{36\cdot 79}{67\cdot 03}$$
$$b^3 = 0\cdot 54885872$$
$$b = 0\cdot 54885872^{\frac{1}{3}}$$
$$b = 0\cdot 81875$$

1: $a(0\cdot 81875)^2 = 67\cdot 03$

$a = 100$

Then

$\underline{A = 10,\ t = ?}$

$$10 = 100(0\cdot 81875)^t$$
$$0\cdot 81875^t = 0\cdot 1$$
$$t = \log_{0\cdot 81875} 0\cdot 1$$
$$t = 11\cdot 51 \text{ minutes.}$$

2. (i) Calculating ratios of successive terms:

$$\frac{9000}{10000} = 0\cdot 9,\ \frac{8100}{9000} = 0\cdot 9,\ \frac{7290}{8100} = 0\cdot 9,\ \frac{6561}{7290} = 0\cdot 9$$

As the ratio is constant, W is an exponential function of x.

(ii) Let $W = a(b)^x$.

Then $b = 0\cdot 9$, and

$$10000 = a(0\cdot 9)^0$$
$$a = 10000$$

Thus

$W = 10000(0\cdot 9)^x$

(iii) When $x = 2\cdot 75$,

$W = 10000(0\cdot 9)^{2\cdot 75}$

$W = 7484\cdot 57$ g

(iv) Half the original amount is 5000 g.

$5000 = 10000(0\cdot 9)^x$

$(0\cdot 9)^x = 0\cdot 5$

$x = \log_{0\cdot 9} 0\cdot 5$

$x = 6\cdot 579$

(v) $W = 1000$:

$1000 = 10000(0\cdot 9)^x$

$(0\cdot 9)^x = 0\cdot 1$

$x = \log_{0\cdot 9} 0\cdot 1$

$x = 21\cdot 854$.

3. (i) $A(t) = Ce^{-bt}$

$A(0) = 1000$: $1000 = Ce^0$

$C = 1000$

$A(4) = 367\cdot 88$: $367\cdot 88 = 1000e^{-b(4)}$

$0\cdot 36788 = e^{-4b}$

$-4b = \ln 0\cdot 36788$

$-4b = -0\cdot 9999$

$b = 0\cdot 25$

Thus $A(t) = 1000e^{-0\cdot 25t}$

and $A(1) = 1000e^{-0\cdot 25} = 778\cdot 8$

$A(2) = 1000e^{-0\cdot 5} = 606\cdot 53$

$A(3) = 1000e^{-0\cdot 75} = 472\cdot 37$

(ii) $A(t) = 50$

$1000e^{-0\cdot 25t} = 50$

$e^{-0\cdot 25t} = 0\cdot 05$

$-0\cdot 25t = \ln 0\cdot 05$

$-0\cdot 25t = -2\cdot 9957$

$t = 11\cdot 98$

(iii) $d > 0\cdot 25$, e.g. $d = 0\cdot 5$.

(iv) Note $B(0) = 1000$, on substituting $t = 0$.

Given: $B(15) = \frac{1}{2}A(15)$

$1000e^{-d(15)} = \frac{1}{2}\left[1000e^{-0\cdot 25(15)}\right]$

$$1000e^{-15d} = 500e^{-3\cdot 75}$$
$$1000e^{-15d} = 11\cdot 75887293$$
$$e^{-15d} = 0\cdot 01175887$$
$$-15d = \ln 0\cdot 01175887$$
$$-15d = -4\cdot 443147$$
$$d = 0\cdot 2962$$

4. (i) $A(t) = Pe^{bt}$

$A(1) = 5724\cdot 46:\quad 5724\cdot 64 = Pe^{b}$ **... 1**

$A(2) = 5958\cdot 08:\quad 5958\cdot 08 = Pe^{2b}$ **... 2**

Dividing **2** by **1**,

$$\frac{Pe^{2b}}{Pe^{b}} = \frac{5958\cdot 08}{5724\cdot 64}$$
$$e^{b} = 1\cdot 04077811$$
$$b = \ln 1\cdot 04077811$$
$$b = 0\cdot 03997$$

1: $5724\cdot 64 = P(1\cdot 04077811)$

$P = 5500\cdot 35$

Thus $A(t) = 5500\cdot 35e^{0\cdot 03997t}$

and $A(0) = P = 5500\cdot 35$

$A(3) = Pe^{3b} = 6201\cdot 04$

$A(4) = Pe^{4b} = 6453\cdot 91$

(ii) $A(t) = 2P$

$$2P = Pe^{0\cdot 03997t}$$
$$2 = e^{0\cdot 03997t}$$
$$0\cdot 03997t = \ln 2$$
$$0\cdot 03997t = 0\cdot 6931471806$$
$$t = 17\cdot 34$$

(iii) $A(t) = P(1+i)^{t}$

$(1+i)^{t} = e^{bt}$

$1+i = e^{b}$

$1+i = 1\cdot 04077811$

$i = 0\cdot 04077811$

5. **(i)** $Q(t) = 200(1\cdot 04)^{t}$

Let $e^{b} = 1\cdot 04$

$b = \ln 1\cdot 04 = 0\cdot 03922071315$

Thus $Q(t) = 200e^{0\cdot 03922071315t}$

(ii) $Q(t) = 400$

$200(1\cdot 04)^{t} = 400$

$(1\cdot 04)^t = 2$

$t = \log_{1\cdot 04} 2 = 17\cdot 67$

(iii) $\dfrac{Q(t+k)}{Q(t)} = \dfrac{200(1\cdot 04)^{t+k}}{200(1\cdot 04)^t} = 1\cdot 04^k$

which is independent of t.

6. **(i)** $Q(t) = Ae^{-bt}$

$Q(1602) = \dfrac{1}{2}A$

$Ae^{-1602b} = \dfrac{1}{2}A$

$e^{-1602b} = 0\cdot 5$

$-1602b = \ln 0\cdot 5$

$-1602b = -0\cdot 6931471806$

$b = 0\cdot 000432676$

and $A = 100$.

(ii) In 4000, $t = 2000$.

$Q(2000) = 100e^{-0\cdot 000432676(2000)}$

$Q(2000) = 42\cdot 09$ grams

(iii) In 3000, $t = 1000$.

$Q(1000) = 100e^{-0\cdot 000432676(1000)}$

$Q(1000) = 64\cdot 88$

The mass of radium that decreases between 3000 and 4000 is

$64\cdot 88 - 42\cdot 09$

$= 22\cdot 79$ grams.

7. Same size:

$P(t) = Q(t)$

$8\cdot 3\times 10^6 \times e^{0\cdot 15t} = 5\cdot 2\times 10^5 \times e^{0\cdot 2t}$

$\dfrac{8\cdot 3\times 10^6}{5\cdot 2\times 10^5} = \dfrac{e^{0\cdot 2t}}{e^{0\cdot 15t}}$

$15\cdot 9615 = e^{0\cdot 05t}$

$0\cdot 05t = \ln 15\cdot 9615$

$0\cdot 05t = 2\cdot 7701$

$t = 55$

8. (i) $f(0) = 500$

$f(1) = f(0) - 4\%$ of $f(0)$

$f(1) = 96\%$ of $f(0)$

$f(1) = f(0)\times 0\cdot 96 = 500\times 0\cdot 96$

(ii) $f(2) = f(1)\times 0\cdot 96$

$f(2) = \left[f(0)\times 0\cdot 96\right]\times 0\cdot 96 = f(0)\times (0\cdot 96)^2$

$f(3) = f(0) \times (0 \cdot 96)^3$

$f(x) = f(0) \times (0 \cdot 96)^x = 500(0 \cdot 96)^x$

(iii) $200 = 500(0 \cdot 96)^x$

$0 \cdot 96^x = 0 \cdot 4$

$x = \log_{0 \cdot 96} 0 \cdot 4$

$x = 22$, correct to the nearest year.

9. (i) $f(0) = N$

$f(1) = N + 3\%$ of N

$f(1) = N(1 \cdot 03)$

$f(2) = N(1 \cdot 03)^2$

$f(t) = N(1 \cdot 03)^t$

(ii) $2N = N(1 \cdot 03)^t$

$1 \cdot 03^t = 2$

$t = \log_{1 \cdot 03} 2$

$t = 23$, correct to the nearest hour

(iii) $143286 = N(1 \cdot 03)^6$

$N = \dfrac{143286}{1 \cdot 03^6}$

$N = 12000$.

10. **(i)** $A(t) = P(1 + 0 \cdot 025)^t$

$A(t) = P(1 \cdot 025)^t$

(ii) $A(t_1) = 1 \cdot 5P$

$P(1 \cdot 025)^{t_1} = 1 \cdot 5P$

$(1 \cdot 025)^{t_1} = 1 \cdot 5$

$t_1 = \log_{1 \cdot 025} 1 \cdot 5$

$t_1 = 16 \cdot 42$

(iii) $A(2t_1) = A(32 \cdot 84)$

$= P(1 \cdot 025)^{32 \cdot 84}$

$= 2 \cdot 25P$

$\neq 2P$

(iv) $A(16 \cdot 42 + t_2) = 2P$

$P(1 \cdot 025)^{16 \cdot 42 + t_2} = 2P$

$1 \cdot 025^{16 \cdot 42 + t_2} = 2$

$16 \cdot 42 + t_2 = \log_{1 \cdot 025} 2$

$t_2 = 11 \cdot 65$.

11. **(i)** $f(t) = 100000(1 - 0 \cdot 08)^t$

$f(t) = 100000(0 \cdot 92)^t$

(ii) $f(t) = 50000$

$100000(0\cdot 92)^t = 50000$

$(0\cdot 92)^t = 0\cdot 5$

$t = \log_{0\cdot 92} 0\cdot 5$

$t = 8\cdot 31$, correct to two decimal places

(iii) $f(t) = 20000$

$100000(0\cdot 92)^t = 20000$

$(0\cdot 92)^t = 0\cdot 2$

$t = \log_{0\cdot 92} 0\cdot 2$

$t = 19\cdot 3$ years.

Revision Exercises 8

1. (i) $2^x.2^{x+1} = 10$

$2^{2x+1} = 10$

$\log_2 10 = 2x + 1$

$2x = \log_2 10 - 1$

$x = \frac{1}{2}\left(\log_2 10 - 1\right)$

(ii) $x = 1\cdot 161$

2. $\frac{8^n \times 2^{2n}}{4^{3n}} = \frac{(2^3)^n \times 2^{2n}}{(2^2)^{3n}}$

$= \frac{2^{5n}}{2^{6n}}$

$= \frac{1}{2^n}$

3. (i) $x + \frac{1}{x} = \frac{10}{3}$ Multiply by $3x$.

$3x^2 + 3 = 10x$

$3x^2 - 10x + 3 = 0$

$(3x-1)(x-3) = 0$

$x = \frac{1}{3}$ or $x = 3$

(ii) $e^y + e^{-y} = \frac{10}{3}$

$e^y + \frac{1}{e^y} = \frac{10}{3}$

From part (i),

$$e^y = \frac{1}{3} \quad \text{or} \quad e^y = 3$$

$$y = \ln\frac{1}{3} \quad \text{or} \quad y = \ln 3$$

4. $2^{2x+1} = 3^x$

$2^x.2^1 = 3^x$

$2 = \dfrac{3^x}{2^x}$

$2 = \left(\dfrac{3}{2}\right)^x$

$2 = 1\cdot5^x$

$x = \log_{1\cdot5} 2$

$x = 1\cdot710$

5. **(i)** $f(x) = 80(1-0\cdot15)^x = 80(0\cdot85)^x$

$f(4) = 80(0\cdot85)^4$

$= 41\cdot76$ mg

(ii) $f(x) = 20$

$80(0\cdot85)^x = 20$

$(0\cdot85)^x = 0\cdot25$

$x = \log_{0\cdot85} 0\cdot25$

$x = 8\cdot53$

(iii) The third hour is from $x = 2$ to $x = 3$.

$f(2) = 80(0\cdot85)^2 = 57\cdot8$

$f(3) = 80(0\cdot85)^3 = 49\cdot13$

Reduction in third hour

$= f(2) - f(3)$

$= 57\cdot8 - 49\cdot13$

$= 8\cdot67$ mg.

6. **(i)** $f(x) = Pe^{-kx}$

$f(0) = Pe^0 = P$

(ii) $f(10000) = 0\cdot5P$

$Pe^{-10000k} = 0\cdot5P$

$e^{-10000k} = 0\cdot5$

$-10000k = \ln 0\cdot5$

$-10000k = -0\cdot6931$

$k = 0\cdot00006931$

$k = 6\cdot931 \times 10^{-5}$

(iii) $f(1000) = Pe^{-(6\cdot 931\times 10^{-5})\times 10^{3}}$

$f(1000) = Pe^{-0\cdot 06931}$

$f(1000) = 0\cdot 9330P$.

7. €P invested now at 4% per annum compound interest in 5 years time will be worth

$€P(1\cdot 04)^5 = €1\cdot 21665P$

If this is to pay for the replacement machine, then

$1\cdot 21665P = 500000$

$P = \dfrac{500000}{1\cdot 21665}$

$P = 410965$

8. **(i)** Let P be the initial size of the colony and $f(x)$ be the size of the colony after x days. Then

$f(x) = P(1\cdot 08)^x$

To double in size:

$f(x) = 2P$

$P(1\cdot 08)^x = 2P$

$(1\cdot 08)^x = 2$

$x = \log_{1\cdot 08} 2$

$x = 9\cdot 01$

Thus it will take $9\cdot 01$ days to double in size.

(ii) To treble in size:

$f(x) = 3P$

$P(1\cdot 08)^x = 3P$

$(1\cdot 08)^x = 3$

$x = \log_{1\cdot 08} 3$

$x = 14\cdot 27$

Thus it will take $14\cdot 27$ days to treble in size.

(iii) $P = 8\times 10^7$ and

$f(30) = [8\times 10^7]\times(1\cdot 08)^{30}$

$f(30) = [8\times 10^7]\times 10\cdot 063$

$f(30) = 8\cdot 05\times 10^8$.

9. $Q(t) = Ae^{bt}$

$Q(10) = 2A$

$Ae^{10b} = 2A$

$e^{10b} = 2$

$10b = \ln 2$

$10b = 0\cdot 6931$

$b = 0\cdot 06931$.